Stability and interconnections of hybrid and impulsive systems

Andrii Mironchenko

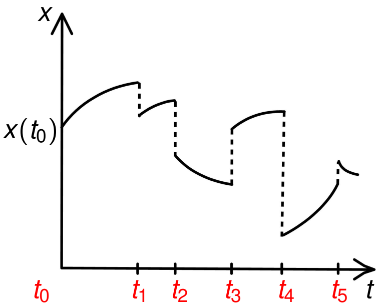
Institute of Mathematics
University of Würzburg

joint work with
Sergey Dashkovskiy, Daniel Liberzon and Guosong Yang

January 07, 2014
Fukuoka, Japan

Outline

1. Impulsive systems
   - Basic definitions
   - Stability theory for single impulsive system
   - Stability theory for interconnected impulsive systems
2. Hybrid systems
3. Conclusion



$$\dot{x}(t) = f(x(t), u(t)) \quad , \; t \notin \{t_1, t_2, \ldots\},$$
$$x(t) = g(x^-(t), u^-(t)) \quad , \; t \in \{t_1, t_2, \ldots\}.$$

$u \in L_{\infty,\, loc}([t_0, \infty), \mathbb{R}^m)$, $x \in \mathbb{R}^n$ is abs. continuous between impulses,
$x^-(t) := \lim_{s \nearrow t} x(s)$, $u^-(t) := \lim_{s \nearrow t} u(s)$.

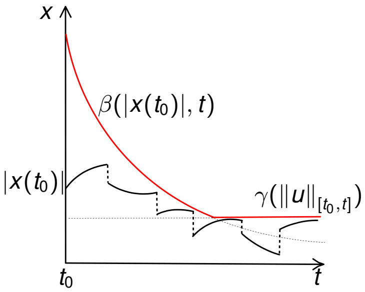
ISS of impulsive system

Definition (Input-to-state stability (ISS))

Σ is ISS for a given impulse time sequence T, if $\exists \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, s.t.

$$|x(t)| \le \max\{\beta(|x(t_0)|, t - t_0), \gamma(\|u\|_{[t_0,t]})\}$$

holds for all $x(t_0) \in \mathbb{R}^n$, $u \in L_{\infty,\ loc}([t_0, \infty), \mathbb{R}^m)$, $t \ge t_0$.
Σ is uniformly ISS w.r.t. a class S of impulse time sequences, if it is ISS $\forall T \in S$, and the functions β and γ do not depend on the choice of $T \in S$.



ISS-Lyapunov functions (ISS-LF)

$$\Sigma: \quad \begin{array}{l} \dot{x}(t) = f(x(t), u(t)), \ t \neq t_k, \\ x(t) = g(x^-(t), u^-(t)), \ t = t_k, \ k \in \mathbb{N}. \end{array}$$

Definition (ISS-Lyapunov function)

A Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an ISS-Lyapunov function for Σ if $\exists \ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that

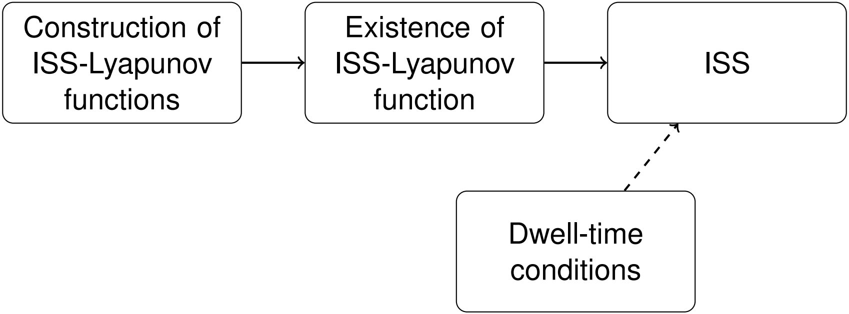
$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), x \in \mathbb{R}^n$$

holds and $\exists \gamma \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$, such that

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x, u) \leq -\varphi(V(x)), & \text{f.a.a. } x, u \\ V(g(x, u)) \leq \alpha(V(x)), \ \forall x, u \end{cases}$$

If $\varphi(x) = cx$ and $\alpha(x) = e^{-d}x$, then V is an exponential ISS-Lyapunov function.

ISS-Theory for impulsive systems



ODE case: $g(x,\cdot) \equiv x$

$$\dot{x}(t) = f(x(t), u(t)) \qquad (1)$$

System (1) is ISS, if $\exists \beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ s.t. $\forall\ x(0) \in \mathbb{R}^n$, $u \in L_{\infty,loc}([0,\infty), \mathbb{R}^m)$

$$|x(t)| \leq \max\{\beta(|x(0)|, t), \gamma(\|u\|_{[0,t]})\}, t \geq 0.$$

Definition (ISS-Lyapunov function)

A Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an ISS-LF for (1) if

- $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that $\psi_1(|x|) \leq V(x) \leq \psi_2(|x|)$, $\forall x \in \mathbb{R}^n$
- $\exists \varphi \in \mathcal{P}$ and $\gamma \in \mathcal{K}_\infty$, such that

$$V(x) \geq \gamma(|u|) \Rightarrow \ \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x))$$

Theorem (Sontag, Wang 1995)

(1) *is ISS* $\Leftrightarrow$ (1) *possesses an ISS-Lyapunov function.*

Exponential ISS-Lyapunov functions

$$\Sigma : \quad \begin{array}{l} \dot{x}(t) = f(x(t), u(t)), \ t \neq t_k, \\ x(t) = g(x^-(t), u^-(t)), \ t = t_k, \ k \in \mathbb{N}. \end{array}$$

Definition

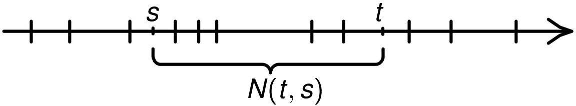
A Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an exponential ISS-Lyapunov function with rate coefficients c, d for Σ if $\exists \ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), x \in \mathbb{R}^n$$

holds and $\exists c, d \in \mathbb{R}$ and $\gamma \in \mathcal{K}_\infty$ such that

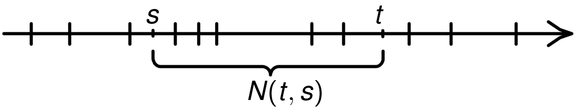
$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x, u) \leq -cV(x) \\ V(g(x, u)) \leq e^{-d} V(x). \end{cases}$$

Average dwell time (ADT) condition



$N(t, s)$ is the number of impulse times t_k in $(s, t]$.

Average dwell time (ADT) condition



$N(t,s)$ is the number of impulse times t_k in $(s,t]$.

Theorem (Hespanha, Liberzon, Teel 2008)

Let V be an exponential ISS-LF for Σ with $c, d \in \mathbb{R}$, $d \neq 0$. For arbitrary $\mu, \lambda > 0$, let $\mathcal{S}[\mu,\lambda]$ denote the class of impulse time sequences $\{t_k\}$ satisfying

$$\text{ADT: } -dN(t,s) - (c-\lambda)(t-s) \le \mu, \quad \text{for any } s,t : 0 \le s \le t.$$

Then Σ is uniformly ISS over $\mathcal{S}[\mu,\lambda]$.

Generalized ADT condition

Theorem (S.D., A.M., SICON 2013)

Let V be an exponential ISS-Lyapunov function for Σ with corresponding coefficients $c \in \mathbb{R}$, $d \neq 0$.
For arbitrary $h : \mathbb{R}_+ \to (0, \infty)$, for which there exist $g \in \mathcal{L}$: $h(x) \leq g(x)$ for all $x \in \mathbb{R}_+$ consider the class $\mathcal{S}[h]$ of impulse time-sequences, satisfying gADT condition:

$$-dN(t,s) - c(t-s) \leq \ln h(t-s), \quad \forall t \geq s \geq t_0. \qquad \text{(gADT)}$$

Then Σ is uniformly ISS over $\mathcal{S}[h]$.

Corollary

Taking $\forall x \in \mathbb{R}_+$ $h(x) = e^{\mu - \lambda x}$, we obtain the ADT condition.

ISS via nonexponential ISS-Lyapunov functions

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x)) \\ V(g(x,u)) \leq \alpha(V(x)). \end{cases}$$

Define $S_\theta := \{\{t_i\}_1^\infty \subset [t_0, \infty) \ : \ t_{i+1} - t_i \geq \theta, \ \forall i \in \mathbb{N}\}$.

Theorem (S.D., A.M., SICON 2013)

Let V be an ISS-Lyapunov function for Σ with $\varphi, \alpha \in \mathcal{P}$. Let for some $\theta, \delta > 0$ and all $a > 0$ it hold nonlinear dwell-time condition

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta. \qquad \text{(nDT)}$$

Then Σ is ISS for all impulse time sequences $T \in S_\theta$.

ISS via nonexponential ISS-Lyapunov functions-2

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x)) \\ V(g(x,u)) \leq \alpha(V(x)). \end{cases}$$

Define $\tilde{S}_\theta := \{\{t_i\}_1^\infty \subset [t_0, \infty) \ : \ t_{i+1} - t_i \leq \theta, \ \forall i \in \mathbb{N}\}$.

Theorem (S.D., A.M., SICON 2013)

Let V be an ISS-Lyapunov function for Σ, α is as in the Definition of ISS-LF and $-\varphi \in \mathcal{P}$. Let for some $\theta, \delta > 0$ and all $a > 0$ it hold

$$\int_{\alpha(a)}^{a} \frac{ds}{-\varphi(s)} \geq \theta + \delta. \qquad (2)$$

Then Σ is ISS w.r.t. every sequence from $\tilde{S}_\theta$.

Example

$$\begin{cases} \dot{x} = & -x^3 + u, \ t \notin T \\ x(t) = & x^-(t) + (x^-(t))^3 + u^-(t), \ t \in T. \end{cases} \quad (3)$$

Take $V : \mathbb{R}^+ \to \mathbb{R}^+$ as $V(x) = |x|$.

The Lyapunov gain χ we choose by $\chi(r) = \left(\frac{r}{a}\right)^{\frac{1}{3}}$, $r \in \mathbb{R}^+$, for some $a \in (0, 1)$.
Condition $|x| \geq \chi(|u|)$ implies

$$\begin{aligned} \dot{V}(x) &\leq -(1-a)(V(x))^3, \\ V(g(x,u)) &\leq V(x) + (1+a)(V(x))^3. \end{aligned}$$

Integral on the lhs of (nDT) takes form

$$I(y,a) = \int_y^{y+(1+a)y^3} \frac{dx}{(1-a)x^3} \leq \frac{1+a}{(1-a)}.$$

$\forall \varepsilon > 0$ there exists a_ε such that $I(y, a_\varepsilon) \leq 1 + 2\varepsilon$.
$\forall \varepsilon > 0$ we can choose $\theta := 1 + \varepsilon$. Note, that the smaller θ we take, the larger are the gains.

What do we need in order to verify ISS of impulsive systems?

What we know till now

What do we need in order to verify ISS of impulsive systems?

- ISS-Lyapunov functions
- Restrictions on the set of impulse time sequences

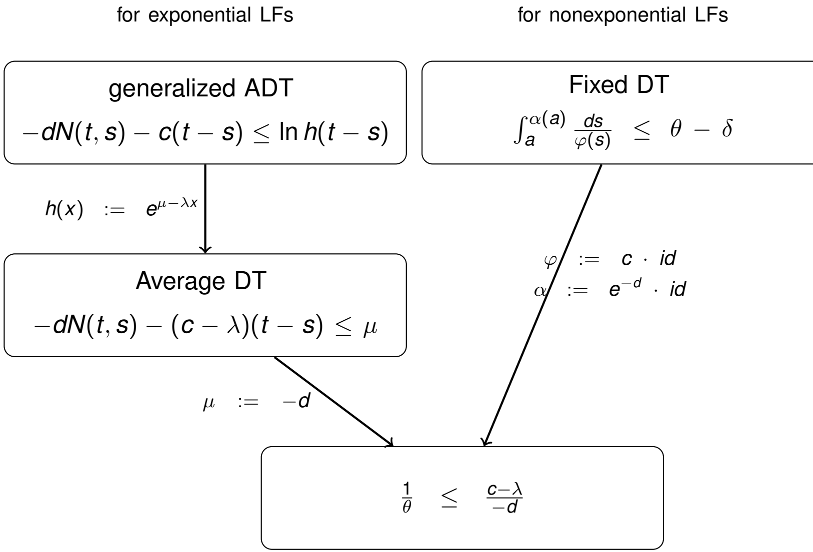
What we know till now

What do we need in order to verify ISS of impulsive systems?

- ISS-Lyapunov functions
- Restrictions on the set of impulse time sequences

What kind of restrictions do we need?

Overview of dwell-time conditions



ISS-Lyapunov functions for subsystems

$$\Sigma : \quad \begin{cases} \dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u(t)), \; t \notin T_i, \\ x_i(t) = g_i(x_1^-(t), \ldots, x_n^-(t), u^-(t)), \; t \in T_i, \\ i = \overline{1, n}, \end{cases}$$

ISS-Lyapunov functions for subsystems

$$\Sigma : \quad \begin{cases} \dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u(t)), \; t \notin T, \\ x_i(t) = g_i(x_1^-(t), \ldots, x_n^-(t), u^-(t)), \; t \in T, \\ i = \overline{1, n}, \end{cases}$$

$V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is an ISS-Lyapunov function for i-th subsystem of Σ if:

1. $\exists \psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$: $\quad \psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|), \quad \forall x_i \in \mathbb{R}^{N_i}$
2. There exist $\gamma_{ij}, \gamma_i \in \mathcal{K}$ and $\varphi_i \in \mathcal{P}$, so that

$$V_i(x_i) \geq \max\{\max_{j=1}^{n} \gamma_{ij}(V_j(x_j)), \gamma_i(|\xi|)\}$$

   implies

$$\nabla V_i \cdot f_i(x, \xi) \leq -\varphi_i\left(V_i(x_i(t))\right).$$

3. There exist $\alpha_i \in \mathcal{P}$, such that

$$V_i(g_i(x, \xi)) \leq \max\{\alpha_i(V_i(x_i)), \max_{j=1}^{n} \gamma_{ij}(V_j(x_j)), \gamma_i(|\xi|)\}.$$

Small-gain theorem

Let $\Gamma_M = (\gamma_{ij})_{i,j=1,\dots,n}$, $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ (gain matrix).
Let us introduce the gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(s) := \left(\max_{j=1}^{n} \gamma_{1j}(s_j), \dots, \max_{j=1}^{n} \gamma_{nj}(s_j) \right), \; s \in \mathbb{R}^n_+.$$

Theorem (Small-gain theorem)

Let for every Σ_i there exist an ISS-Lyapunov function V_i with corresponding gains γ_{ij}. If the small-gain condition

$$\Gamma(s) \not\geq s, \; \forall\, s \in \mathbb{R}^n_+ \setminus \{0\},$$

holds, then Σ possesses an ISS-Lyapunov function, defined by

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}.$$

Small-gain theorem for construction of eISS LFs

Let

$$P := \{f : \mathbb{R}_+ \to \mathbb{R}_+ : \exists a \geq 0,\ b > 0 : \ f(s) = as^b \ \forall s \in \mathbb{R}_+\}$$

Theorem (SGT when subsystems have exponential ISS-LF)

Let V_i be eISS Lyapunov function for Σ_i with corresponding gains γ_{ij}, $i = 1, \ldots, n$, for which the small-gain condition holds. Let also $\gamma_{ij} \in P$. Then function $V : \mathbb{R}^N \to \mathbb{R}_+$, defined by

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\},$$

is an eISS Lyapunov function for Σ for certain σ.

E.g. $\sigma = Q(at)$, $a > 0$, $Q(x) := MAX\{x, \Gamma(x), \Gamma^2(x), \ldots, \Gamma^{n-1}(x)\}$
(due to I. Karafyllis, Z.-P. Jiang (IMA Journal of Math. Cont. and Inf., 2011))

Small-gain contra Dwell-time

Interconnected system

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + x_2^2(t),\ t \notin T,\\ x_1(t) &= e^{-1}x_1^-(t),\ t \in T\end{aligned}$$

and

$$\begin{aligned}\dot{x}_2(t) &= -x_2(t) + 3\sqrt{|x_1(t)|},\ t \notin T,\\ x_2(t) &= e^{-1}x_2^-(t),\ t \in T.\end{aligned}$$

Small-gain contra Dwell-time

Interconnected system

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t), \; t \notin T,$$
$$x_1(t) = e^{-1} x_1^-(t), \; t \in T$$

and

$$\dot{x}_2(t) = -x_2(t) + 3\sqrt{|x_1(t)|}, \; t \notin T,$$
$$x_2(t) = e^{-1} x_2^-(t), \; t \in T.$$

ISS-Lyapunov functions and gains

$$V_1(x_1) = |x_1|, \quad \gamma_{12}(r) = \frac{1}{a} r^2,$$
$$V_2(x_2) = |x_2|, \quad \gamma_{21}(r) = \frac{1}{b}\sqrt{r},$$

Small-gain contra Dwell-time

Interconnected system

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t),\ t \notin T,$$
$$x_1(t) = e^{-1} x_1^-(t),\ t \in T$$

and

$$\dot{x}_2(t) = -x_2(t) + 3\sqrt{|x_1(t)|},\ t \notin T,$$
$$x_2(t) = e^{-1} x_2^-(t),\ t \in T.$$

ISS-Lyapunov functions and gains

$$V_1(x_1) = |x_1|, \quad \gamma_{12}(r) = \tfrac{1}{a} r^2,$$
$$V_2(x_2) = |x_2|, \quad \gamma_{21}(r) = \tfrac{1}{b}\sqrt{r},$$

Estimates for derivatives of LFs

$$|x_1| \geq \gamma_{12}(|x_2|) \Rightarrow \dot{V}_1(x_1) \leq (a-1) V_1(x_1),$$
$$|x_2| \geq \gamma_{21}(|x_1|) \Rightarrow \dot{V}_2(x_2) \leq (3b-1) V_2(x_2).$$

Small-gain contra Dwell-time

The small-gain condition

$$\gamma_{12} \circ \gamma_{21}(r) = \frac{1}{ab^2} r < r, \quad \forall r > 0$$

ISS-Lyapunov function for interconnection

$$V(x) = \max\{|x_1|, c^2|x_2|^2\}, \quad \text{where } \frac{1}{b} < \frac{1}{c} < \sqrt{a}$$

Estimate for $\dot{V}(x)$ and $V(g(x))$

$$\frac{d}{dt} V(x) \leq \max\{(a-1), 2(3b-1)\} V(x).$$

$$V(g(x)) = V(e^{-1} \cdot x) \leq e^{-1} V(x).$$

Choose a, b so that $(a-1) = 2(3b-1)$ and small-gain condition holds. Then choose any c as above.

Summary

We presented

- Sufficient conditions of Lyapunov type for ISS of impulsive systems, based on
  1. nonlinear dwell-time conditions
  2. generalized average DT condition
- Construction of ISS-Lyapunov function via small-gain theorems.
  - exponential
  - nonexponential

The constructions work only if instabilities are matched!

Outline

1 Impulsive systems
- Basic definitions
- Stability theory for single impulsive system
- Stability theory for interconnected impulsive systems

2 Hybrid systems

3 Conclusion

Hybrid systems

$$\Sigma : \begin{array}{ll} \dot{x} \in f(x,u), & (x,u) \in C, \\ x^+ \in g(x,u), & (x,u) \in D, \end{array}$$

Hybrid time domain

$$\text{dom} := \cup_{k=1}^{T} [t_k, t_{k+1}] \times \{k\},$$

$\{t_i\}$ is nondecreasing (finite or infinite) sequence.

- A hybrid signal is a function defined on a hybrid time domain.
- A hybrid input is a hybrid signal $u : \text{dom}\, u \to \mathbb{R}^m$, so that $u(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j.
- A hybrid arc is a hybrid signal $x : \text{dom}\, x \to X$, which is locally absolutely continuous on each interval of $\text{dom}\, x$ with nonempty interior.

Solution concept

$$\Sigma : \begin{array}{ll} \dot{x} \in f(x,u), & (x,u) \in C, \\ x^+ \in g(x,u), & (x,u) \in D, \end{array}$$

Let $x(t,j)$ be the state of Σ after t time units and j jumps.

A pair (x,u) is a solution pair of Σ if:

- $\operatorname{dom} x = \operatorname{dom} u$;
- for all $j \in \mathbb{N}$ and almost all $t \in \mathbb{R}$ so that $(t,j) \in \operatorname{dom} x$ $(x(t,j),u(t,j)) \in C$ and $\dot{x}(t,j) \in f(x(t,j),u(t,j))$;
- for all $(t,j) \in \operatorname{dom} x$ so that $(t,j+1) \in \operatorname{dom} x$ $(x(t,j),u(t,j)) \in D$ and $x(t,j+1) \in g(x(t,j),u(t,j))$.

ISS of hybrid systems

$$\Sigma : \begin{array}{ll} \dot{x} \in f(x,u), & (x,u) \in C, \\ x^+ \in g(x,u), & (x,u) \in D, \end{array}$$

A set of solution pairs $\mathcal{S}$ of Σ is
pre-input-to-state stable (pre-ISS) w.r.t. $\mathcal{A} \subset X$ if
$\exists \beta \in \mathcal{KL},\ \gamma \in \mathcal{K}_\infty$: $\quad \forall (x,u) \in \mathcal{S}, \forall (t,j) \in \operatorname{dom} x$

$$|x(t,j)|_{\mathcal{A}} \le \max\left\{\beta(|x(0,0)|_{\mathcal{A}}, t+j), \gamma(\|u\|_{(t,j)})\right\}.$$

Σ is pre-ISS w.r.t. $\mathcal{A}$:= $\mathcal{S}$ contains all solution pairs (x,u) of Σ.
Σ is ISS w.r.t. $\mathcal{A}$ if all solutions are in addition complete.

ISS Lyapunov functions for hybrid systems

$$\Sigma : \begin{array}{rl} \dot{x} \in f(x,u), & (x,u) \in C, \\ x^+ \in g(x,u), & (x,u) \in D, \end{array}$$

$V : X \to \mathbb{R}_+$ is an ISS-Lyapunov function for Σ w.r.t. $\mathcal{A} \subset X$ if $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(|x|_{\mathcal{A}}) \le V(x) \le \psi_2(|x|_{\mathcal{A}}) \qquad \forall x \in X$$

and $\exists \chi \in \mathcal{K}_\infty$, $\alpha, \varphi \in \mathcal{P}$:

$$V(x) \ge \chi(|u|) \Rightarrow \begin{cases} \dot{V}(x; y) \le -\varphi(V(x)) & \forall (x,u) \in C,\ y \in f(x,u), \\ V(y) \le \alpha(V(x)) & \forall (x,u) \in D,\ y \in g(x,u), \end{cases}$$

where $\dot{V}(x; y) = \overline{\lim\limits_{h \to +0}} \frac{V(x+hy) - V(x)}{h}$.

Restatement of a Def of ISS Lyapunov function

Lemma

$V : X \to \mathbb{R}_+$ is an ISS-Lyapunov function for Σ w.r.t. $\mathcal{A}$ iff $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(|x|_\mathcal{A}) \leq V(x) \leq \psi_2(|x|_\mathcal{A}) \qquad \forall x \in X$$

and $\exists \chi \in \mathcal{K}_\infty$, $\alpha, \varphi \in \mathcal{P}$ such that $\forall (x,u) \in C, \forall y \in f(x,u)$

$$V(x) \geq \chi(|u|) \Rightarrow \dot{V}(x; y) \leq -\varphi(V(x))$$

and $\forall (x,u) \in D, \forall y \in g(x,u)$ we have

$$V(y) \leq \max\{\alpha(V(x)), \chi(|u|)\}.$$

ADT for hybrid systems

Proposition

Let V be an exponential ISS-Lyapunov function for Σ w.r.t. $\mathcal{A}$ with rate coefficients $c \in \mathbb{R}$, $d \neq 0$.
For arbitrary $\mu \geq 1$ and all $\eta, \lambda > 0$ consider
$\mathcal{S}[\eta, \lambda, \mu]$:= set of solution pairs (x, u) of Σ, satisfying:

$$-(d-\eta)(j-i)-(c-\lambda)(t-s) \leq \mu \quad \forall (t,j),(s,i) \in \operatorname{dom} x.$$

Then $\mathcal{S}[\eta, \lambda, \mu]$ is pre-ISS w.r.t. $\mathcal{A}$.

Interconnections of hybrid systems

$$\Sigma: \begin{array}{ll} \dot{x}_i \in f_i(x_1,\ldots,x_n,u), & (x_1,\ldots,x_n,u) \in C, \\ x_i^+ \in g_i(x_1,\ldots,x_n,u), & (x_1,\ldots,x_n,u) \in D, \\ i = 1,\ldots,n. & \end{array}$$

Theorem

Let V_i be an ISS-Lyapunov function for Σ_i w.r.t $\mathcal{A}_i \subset X_i$ with gains $\chi_{ij}, \chi_i \in \mathcal{K}$. If Γ satisfies the small-gain condition, then

$$V(x) := \max_{i=1}^{n} \sigma_i^{-1}(V_i(x_i)),$$

is an ISS Lyapunov function for Σ w.r.t. $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$, where σ is a smooth Ω-path w.r.t. Γ.

Let $V_i : X_i \to \mathbb{R}_+$ satisfies:

1. There exist $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that

$$\psi_{i1}(|x_i|_{\mathcal{A}_i}) \leq V_i(x_i) \leq \psi_{i2}(|x_i|_{\mathcal{A}_i}) \quad \forall x_i \in X_i.$$

2. There exist $c_i \in \mathbb{R}$, $\chi_{ij}, \chi_i \in \mathcal{K}$, $\chi_{ii} := 0$, and for all $(x, u) \in C$ and all $y_i \in f_i(x, u)$

$$V_i(x_i) \geq \max\left\{\max_{j=1}^{n} \chi_{ij} V_j(x_j), \chi_i(|u|)\right\} \Rightarrow \dot{V}_i(x_i; y_i) \leq -c_i(V_i(x_i)).$$

3. $\exists d_i \in \mathbb{R}$: for all $(x, u) \in D$ and all $y_i \in g_i(x, u)$

$$V_i(y_i) \leq \max\left\{e^{-d_i} V_i(x_i), \max_{j=1}^{n} \chi_{ij} V_j(x_j), \chi_i(|u|)\right\}.$$

Hybrid systems, which model the clocks

Define $I_d := \{i \in \{1, \dots, n\} : d_i < 0\}$.

Our aim: to change the subsystems with indices from I_d

Pick any solution pair (x, u) of Σ and let $(t, j), (s, k) \in \operatorname{dom} x$. For $i \in I_d$ restrict the density of jumps of Σ_i by

$$j - k \leq \delta_i (t - s) + N_0^i, \qquad \text{(ADT)}$$

where $\delta_i, N_0^i > 0$.
Hybrid time domain satisfies (ADT) iff it is the domain of some solution of

$$\begin{aligned} \dot{\tau}_i &\in [0, \delta_i], & \tau_i &\in [0, N_0^i], \\ \tau_i^+ &= \tau_i - 1, & \tau_i &\in [1, N_0^i]. \end{aligned} \qquad (4)$$

Modification of subsystems for $i \in I_d$

$$\tilde{\Sigma}_i : \quad \begin{aligned} \dot{z}_i &\in \tilde{f}_i(z,u), & z &\in \tilde{C}, \\ z_i^+ &\in \tilde{g}_i(z,u), & z &\in \tilde{D}. \end{aligned}$$

$$\tilde{f}_i(z,u) := \begin{pmatrix} f_i(x,u) \\ [0,\delta_i] \end{pmatrix}, \quad \tilde{g}_i(z,u) := \begin{pmatrix} g_i(x,u) \\ \{\tau_i - 1\} \end{pmatrix}.$$

Modified Lyapunov function for $\tilde{\Sigma}_i$:

$$W_i(z_i) = \begin{cases} V_i(x_i), & i \notin I_d, \\ e^{L_i \tau_i} V_i(x_i), & i \in I_d \end{cases} \tag{5}$$

for some $L_i > 0$.

Proposition: W_i is an exponential ISS Lyapunov function

1. There exist $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ such that

$$\tilde{\psi}_{i1}(|z_i|_{\tilde{\mathcal{A}}_i}) \leq W_i(z_i) \leq \tilde{\psi}_{i2}(|z_i|_{\tilde{\mathcal{A}}_i}) \quad \forall z_i \in Z_i.$$

2. For all $(z, u) \in \tilde{C}$ and all $y_i \in \tilde{f}_i(z, u)$,

$$W_i(z_i) \geq \max\left\{\max_{j=1}^{n} \tilde{\chi}_{ij} W_j(z_j), \tilde{\chi}_i |u|\right\} \Rightarrow \dot{W}_i(z_i; y_i) \leq -\tilde{c}_i W_i(z_i),$$

where $\tilde{c}_i = c_i$ for $i \notin I_d$ and $\tilde{c}_i = c_i - L_i\delta_i$ for $i \in I_d$ and

$$\begin{cases} \tilde{\chi}_i := \chi_i,\ \tilde{\chi}_{ij} := \chi_{ij} & i \notin I_d, \\ \tilde{\chi}_i := e^{L_i N_0^i}\chi_i,\ \tilde{\chi}_{ij} := e^{L_i N_0^i}\chi_{ij} & i \in I_d. \end{cases}$$

3. For all $(z, u) \in \tilde{D}$ and all $y_i \in \tilde{g}_i(z, u)$. it holds

$$W_i(y_i) \leq \max\left\{e^{-\tilde{d}_i} W_i(z_i), \max_{j=1}^{n} \tilde{\chi}_{ij} W_j(z_j), \tilde{\chi}_i |u|\right\},$$

General scheme

1. For all $i \in \{1, \ldots, n\}$, construct an exponential ISS Lyapunov function V_i for Σ_i with linear internal gains and rate coefficients c_i, d_i.
2. Compute I_d, I_c.
3. Modify Σ_i *either for all* $i \in I_d$ *or for all* $i \in I_c$.
4. Using small-gain theorem to construct an exponential ISS Lyapunov function W for $\tilde{\Sigma}$ with rate coefficients c, d.
5. Obtain the conditions on ISS of $\tilde{\Sigma}$
6. Obtain the conditions on ISS of the original system Σ.

- It works for interconnections with unmatched instabilities
- Only subsystems from either I_d or I_c should be modified.

Outlook

Abstract ∞-dim systems

Using framework from

- S. Dashkovskiy, A.M. Input-to-state stability of infinite-dimensional control systems, MCSS, 2013.

one can investigate a system

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)) \quad , \; t \notin \{t_1, t_2, \dots\},$$
$$x(t) = g(x^-(t), u^-(t)) \quad , \; t \in \{t_1, t_2, \dots\}.$$

Impulsive systems with time-delays

$$\dot{x}(t) = f(x_t, u(t)) \quad , \; t \notin \{t_1, t_2, \dots\},$$
$$x(t) = g(x_t^-, u^-(t)) \quad , \; t \in \{t_1, t_2, \dots\}.$$

- W.-H. Chen and W. X. Zheng. Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays. Automatica, 2009.
- S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok. Stability of interconnected impulsive systems with and without time-delays using Lyapunov methods. Nonlinear Analysis: Hybrid Systems, 2012.

Possible directions of future work

- Interconnections of hybrid systems with nonexponential Lyapunov functions
- Is it possible to resign from the modification of ISS Lyapunov functions?
- Study of stability of impulsive time-delay systems

Possible directions of future work

- Interconnections of hybrid systems with nonexponential Lyapunov functions
- Is it possible to resign from the modification of ISS Lyapunov functions?
- Study of stability of impulsive time-delay systems

Thank you for attention!