Lyapunov methodology for stability analysis of impulsive systems

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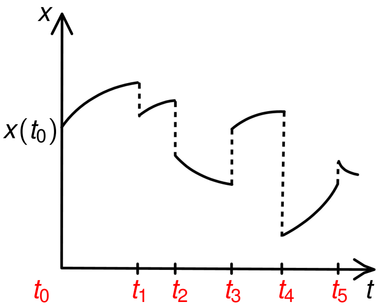
Impulsive systems



$$\dot{x}(t) = f(x(t), u(t)) \quad , \; t \notin \{t_1, t_2, \dots\},$$
$$x(t) = g(x^-(t), u^-(t)) \quad , \; t \in \{t_1, t_2, \dots\}.$$

$u \in L_{\infty,\, loc}([t_0, \infty), \mathbb{R}^m)$, $x \in \mathbb{R}^n$ is abs. continuous between impulses,
$x^-(t) := \lim_{s \nearrow t} x(s)$, $u^-(t) := \lim_{s \nearrow t} u(s)$.

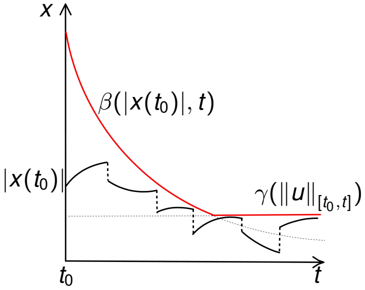
ISS of impulsive system

Definition (Input-to-state stability (ISS))

Σ is ISS for a given impulse time sequence T, if $\exists \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, s.t.

$$|x(t)| \leq \max\{\beta(|x(t_0)|, t - t_0), \gamma(\|u\|_{[t_0,t]})\}$$

holds for all $x(t_0) \in \mathbb{R}^n$, $u \in L_{\infty, loc}([t_0, \infty), \mathbb{R}^m)$, $t \geq t_0$.
Σ is uniformly ISS w.r.t. a class S of impulse time sequences, if Σ is ISS $\forall T \in S$, and β and γ do not depend on the choice of $T \in S$.



ISS-Lyapunov functions (ISS-LF)

$$\Sigma : \quad \begin{array}{l} \dot{x}(t) = f(x(t), u(t)),\ t \neq t_k, \\ x(t) = g(x^-(t), u^-(t)),\ t = t_k,\ k \in \mathbb{N}. \end{array}$$

Definition (ISS-Lyapunov function)

V is an ISS-Lyapunov function for Σ if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$:

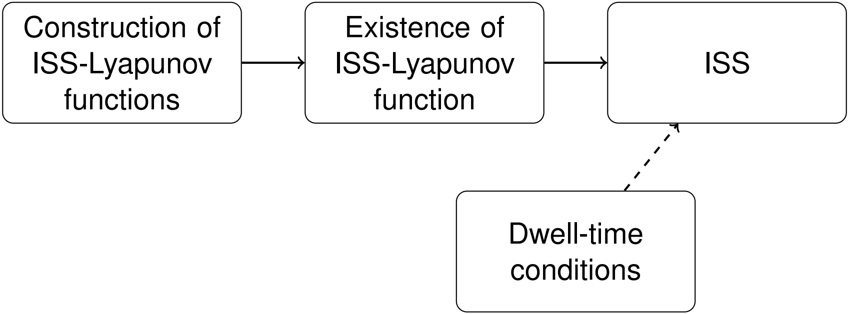
$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), x \in \mathbb{R}^n$$

and $\exists \gamma \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$:

$$V(x) \geq \gamma(|u|) \Rightarrow \left\{ \begin{array}{l} \nabla V(x) \cdot f(x, u) \leq -\varphi(V(x)), \quad \text{f.a.a. } x, u \\ V(g(x, u)) \leq \alpha(V(x)),\ \forall x, u \end{array} \right.$$

If $\varphi(x) = cx$ and $\alpha(x) = e^{-d}x$, then V is an exponential ISS-Lyapunov function.

ISS-Theory for impulsive systems



Exponential ISS-Lyapunov functions

$$\Sigma: \quad \begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \ t \neq t_k, \\ x(t) &= g(x^-(t), u^-(t)), \ t = t_k, \ k \in \mathbb{N}. \end{aligned}$$

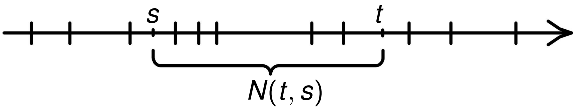
Definition

V is an exponential ISS-LF for Σ if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$:

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), x \in \mathbb{R}^n$$

and $\exists c, d \in \mathbb{R}$ and $\gamma \in \mathcal{K}_\infty$:

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -cV(x) \\ V(g(x,u)) \leq e^{-d} V(x). \end{cases}$$



$N(t,s)$ is the number of impulse times t_k in $(s,t]$.

Generalized ADT condition

Theorem (S.D., A.M., SICON 2013)

Let V be an exponential ISS-LF for Σ with coefficients $c, d \in \mathbb{R}$, $d \neq 0$.
$\forall\ h : \mathbb{R}_+ \to (0,\infty)$*:* $\exists g \in \mathcal{L}$*:* $h(x) \leq g(x)\ \forall x \in \mathbb{R}_+$
$\mathcal{S}[h]$*:*$\Leftrightarrow$ *class of impulse time sequences:*

$$-dN(t,s) - c(t-s) \leq \ln h(t-s) \quad \forall t \geq s \geq t_0. \qquad \text{(gADT)}$$

Then Σ is uniformly ISS w.r.t. $\mathcal{S}[h]$.

Corollary (Hespanha, Liberzon, Teel 2008)

gADT with $h(x) = e^{\mu - \lambda x}$, $x \in \mathbb{R}_+$ $\quad \Rightarrow \quad$ ADT condition.

ADT: $-dN(t,s) - (c-\lambda)(t-s) \leq \mu, \quad$ *for any* $s,t : 0 \leq s \leq t$*.*

ISS via nonexponential ISS-Lyapunov functions

We assume: continuous dynamics is ISS ($\varphi \in \mathcal{P}$)

$$V(x) \geq \gamma(\|u\|_U) \Rightarrow \begin{cases} \dot{V}_u(x) \leq -\varphi(V(x)) \\ V(g(x,u)) \leq \alpha(V(x)). \end{cases}$$

Let $S_\theta := \left\{\{t_i\}_1^\infty \subset [t_0, \infty) \ : \ t_{i+1} - t_i \geq \theta, \ \forall i \in \mathbb{N}\right\}$.

Theorem (S. Dashkovskiy, A.M., SICON, 2013)

Let V be ISS-Lyapunov function for Σ.

$$\exists \theta, \delta > 0 : \quad \int_r^{\alpha(r)} \frac{ds}{\varphi(s)} \leq \theta - \delta, \quad \forall r > 0 \qquad \text{(FDT)}$$

$\Rightarrow$ *ISS $\forall$ impulse time sequences $T \in S_\theta$.*

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- ISS-Lyapunov functions
- Restrictions on the set of impulse time sequences

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What kind of restrictions do we need?

Overview of dwell-time conditions

for exponential LFs

for nonexponential LFs

generalized ADT

$$-dN(t,s) - c(t-s) \leq \ln h(t-s)$$

Fixed DT

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta$$

$h(x) := e^{\mu - \lambda x}$

Average DT

$$-dN(t,s) - (c-\lambda)(t-s) \leq \mu$$

$\varphi := c \cdot id$

$\alpha := e^{-d} \cdot id$

$\mu := -d$

$$\frac{1}{\theta} \leq \frac{c-\lambda}{-d}$$

ISS-Lyapunov functions for subsystems

$$\Sigma : \quad \begin{cases} \dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u(t)), \ t \notin T, \\ x_i(t) = g_i(x_1^-(t), \ldots, x_n^-(t), u^-(t)), \ t \in T, \\ i = \overline{1, n}, \end{cases}$$

ISS-Lyapunov functions for subsystems

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$V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is an ISS-Lyapunov function for i-th subsystem of Σ if:

1. $\exists \psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$: $\quad \psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|), \quad \forall x_i \in \mathbb{R}^{N_i}$
2. There exist $\gamma_{ij}, \gamma_i \in \mathcal{K}$ and $\varphi_i \in \mathcal{P}$, so that

$$V_i(x_i) \geq \max\{\max_{j=1}^{n} \gamma_{ij}(V_j(x_j)), \gamma_i(|\xi|)\}$$

   implies

$$\nabla V_i \cdot f_i(x, \xi) \leq -\varphi_i\left(V_i(x_i(t))\right).$$

3. There exist $\alpha_i \in \mathcal{P}$, such that

$$V_i(g_i(x, \xi)) \leq \max\{\alpha_i(V_i(x_i)), \max_{j=1}^{n} \gamma_{ij}(V_j(x_j)), \gamma_i(|\xi|)\}.$$

Small-gain theorem

Let $\Gamma_M = (\gamma_{ij})_{i,j=1,\ldots,n}$, $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ (gain matrix).
Let us introduce the gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(s) := \left(\max_{j=1}^{n} \gamma_{1j}(s_j), \ldots, \max_{j=1}^{n} \gamma_{nj}(s_j) \right), \; s \in \mathbb{R}^n_+.$$

Theorem (Small-gain theorem)

Let $\forall \Sigma_i$ $\exists$ an ISS-Lyapunov function V_i with gains γ_{ij}. If the small-gain condition

$$\Gamma(s) \not\geq s, \; \forall \, s \in \mathbb{R}^n_+ \setminus \{0\},$$

holds, then Σ possesses an ISS-Lyapunov function, defined by

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}.$$

Small-gain theorem for construction of eISS LFs

$$P := \{f : \mathbb{R}_+ \to \mathbb{R}_+ : \exists a \geq 0,\ b > 0 :\ f(s) = as^b\ \forall s \in \mathbb{R}_+\}$$

Theorem (SGT when subsystems have exponential ISS-LF)

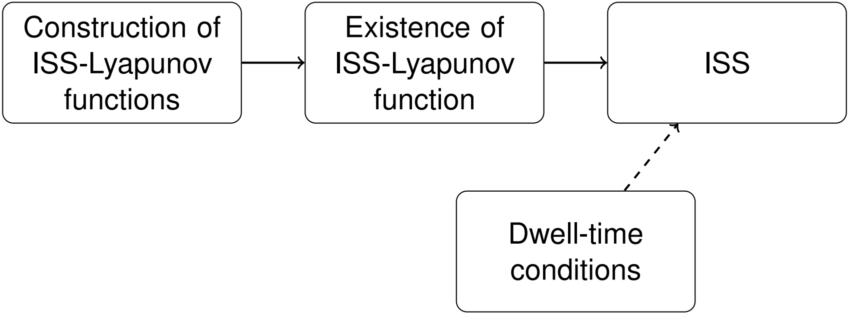
Let V_i be eISS Lyapunov function for Σ_i with gains $\gamma_{ij} \in P$, $i = 1, \ldots, n$, satisfying the small-gain condition. Then

$$V : x \mapsto \max_i \{\sigma_i^{-1}(V_i(x_i))\},$$

is an eISS Lyapunov function for Σ for certain σ.

E.g. $\sigma = Q(at)$, $a > 0$, $Q(x) := MAX\{x, \Gamma(x), \Gamma^2(x), \ldots, \Gamma^{n-1}(x)\}$
(due to I. Karafyllis, Z.-P. Jiang (IMA Journal of Math. Cont. and Inf., 2011))

Summary



- Dwell-time conditions:
  1. nonlinear dwell-time conditions
  2. generalized average DT condition
- Exponential and nonexponential small-gain theorems.

The constructions work only if instabilities are matched!

Outlook

Abstract ∞-dim systems

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)) \quad , \; t \notin \{t_1, t_2, \ldots\},$$
$$x(t) = g(x^-(t), u^-(t)) \quad , \; t \in \{t_1, t_2, \ldots\}.$$

- S. Dashkovskiy, A.M. Input-to-state stability of nonlinear impulsive systems, SICON, 2013.

Impulsive systems with time-delays

- S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok. Stability of interconnected impulsive systems with and without time-delays using Lyapunov methods. Nonlinear Analysis: Hybrid Systems, 2012.

Hybrid systems

- Andrii Mironchenko, Guosong Yang and Daniel Liberzon. Lyapunov small-gain theorems for not necessarily ISS hybrid systems, submitted to MTNS 2014.