Robust stability of interconnections of infinite-dimensional systems: an ISS approach

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Outline

Class of systems

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \ x(t) \in D(A) \subset X, u(t) \in U, \\ x(0) = \phi_0. \end{cases}$$

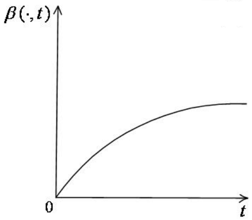
- X = State space
- $U_c = C(\mathbb{R}_+, U)$.
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.
- $f(0,0) = 0$.

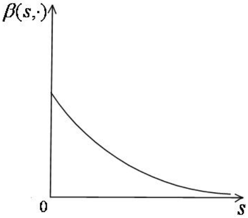
$x \in C([0,T], X)$ is a mild solution iff

$$x(t) = T(t)\phi_0 + \int_0^t T(t-s)f(x(s), u(s))ds.$$

Comparison functions

$$
\begin{aligned}
\mathcal{K}_\infty &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma(0) = 0,\ \gamma \text{ is continuous, growing and unbounded}\} \\
\mathcal{L} &:= \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly decreasing and } \lim_{t\to\infty} \gamma(t) = 0\right\} \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid\ \beta(\cdot, t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r, \cdot) \in \mathcal{L},\ \forall r > 0\}
\end{aligned}
$$





Input-to-state stability

Definition (GAS uniform w.r.t. state (0-UGASx))

0-UGASx $\quad :\Leftrightarrow \quad \exists \beta \in \mathcal{KL}: \quad \forall \phi_0 \in X, \forall t \geq 0$

$$\|\phi(t, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t).$$

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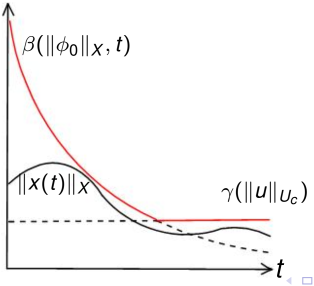
Definition (ISS)

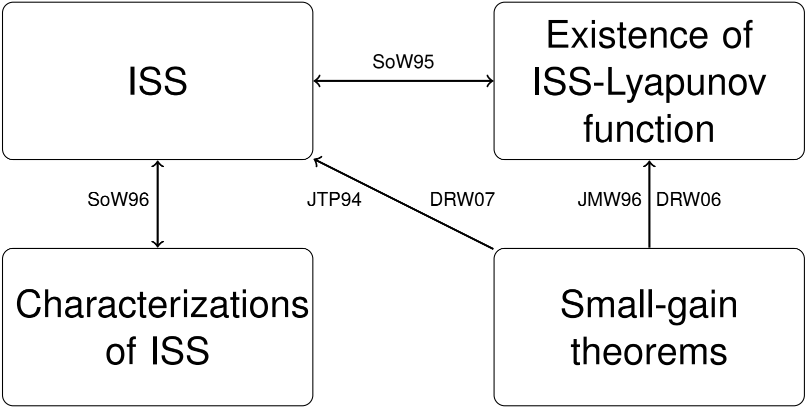
ISS w.r.t. U_c $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$: $\forall t \geq 0, \forall \phi_0 \in X, \forall u \in U_c$

$$\|\phi(t, \phi_0, u)\|_X \leq \max\left\{\beta(\|\phi_0\|_X, t), \gamma(\|u\|_{U_c})\right\}.$$

ISS $:\Leftrightarrow$ ISS w.r.t. $C(\mathbb{R}_+, U)$.



Fundamentals of ISS-Theory for ODEs



Stability concepts for ∞-dim systems

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, B \in L(U, X). \tag{Lin}$$

$$\phi(t, x_0, u) = T(t)x_0 + \int_0^t T(t-r)Bu(r)dr,$$

Fact

0-UGASx $\quad \Leftrightarrow \quad \exists M, \lambda > 0 : \|T(t)\| \leq Me^{-\lambda t} \Leftrightarrow T$ exp. stable .

Fact

0-GAS $\quad \Leftrightarrow \quad \lim_{t\to\infty} \|T(t)x\| = 0 \ \forall x \in X \Leftrightarrow T$ strongly stable.

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Fact

0-UGASx $\Leftrightarrow$ $\exists M, \lambda > 0 : \|T(t)\| \leq Me^{-\lambda t}$ $\Leftrightarrow$ T exp. stable .

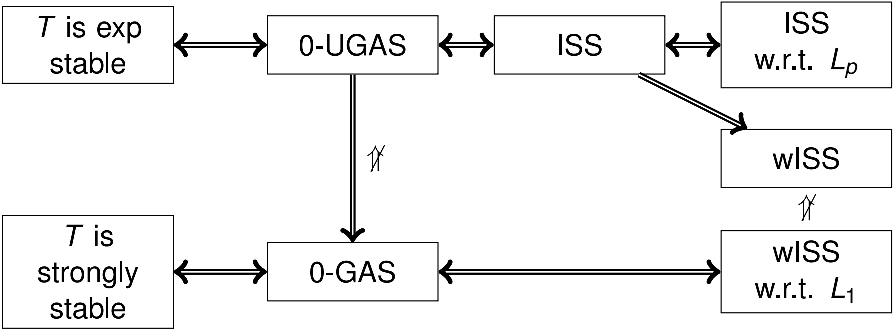
Fact

0-GAS $\Leftrightarrow$ $\lim_{t\to\infty} \|T(t)x\| = 0 \; \forall x \in X$ $\Leftrightarrow$ T strongly stable.

For ∞-dim systems: GAS $\neq$ 0-UGASx.

ISS theory for linear systems with bounded input operators

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, B \in L(U, X). \qquad \text{(Lin)}$$



2 Examples

System with an unbounded input operator

$$\dot{x}(l,t) = -x(l,t) + \tan^{\frac{1}{8}}(l)u(l,t), \quad l \in (0,\pi/2).$$

- This system is 0-UGAS.
- It is not ISS for $X = U = C(0,\pi/2)$.
- It is ISS for $X = L_2(0,\pi/2)$ and $U = L_4(0,\pi/2)$.

0-GAS system

$$\dot{x}(l,t) = -\frac{1}{1+|l|}x(l,t) + u(l,t), \quad l \in \mathbb{R}.$$

- For $U = X = C_0(\mathbb{R})$ the system is 0-GAS, but not 0-UGAS.
- It is not ISS.

Lyapunov functions

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \; x(t) \in D(A) \subset X, u(t) \in U, \\ x(0) = \phi_0. \end{cases}$$

Definition

$V : X \to \mathbb{R}_+$ is ISS-Lyapunov function iff $\exists \psi_1, \psi_2, \chi, \alpha \in \mathcal{K}_\infty$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $V(x) \geq \chi(\|\xi\|_U) \quad \Rightarrow \quad \dot{V}_u(x) \leq -\alpha(V(x)),$

$\forall x \in X, \; \forall \xi \in U, \; \forall u \in U_c$ with $u(0) = \xi$.
Here $\dot{V}_u(x) = \overline{\lim\limits_{t \to +0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x))$.

Theorem

$\exists$ *ISS-Lyapunov function* $\Rightarrow$ *ISS.*

Interconnected systems

$$\Sigma : \begin{cases} \Sigma_i : \quad \dot{x}_i = A_i x_i + f_i(x_1, \ldots, x_n, u), \;\; x_i \in X_i \\ i = 1, \ldots, n \end{cases}$$

- X_i state space of Σ_i
- A_i infinitesimal generator of C_0-semigroup on X_i.
- $X = X_1 \times \ldots \times X_n$ state space of the whole system.
- $\tilde{X}_i := X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n \times U$
  space of inputs into i-th subsystem.

ISS-Lyapunov functions for subsystems

$$\Sigma : \begin{cases} \Sigma_i : \quad \dot{x}_i = A_i x_i + f_i(x_1, \ldots, x_n, u), \ x_i \in X_i \\ i = 1, \ldots, n \end{cases}$$

ISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is ISS-Lyapunov function for Σ_i iff
$\exists \psi_{i1}, \ \psi_{i2}, \ \alpha_i, \ \chi_i, \ \chi_{ij} \in \mathcal{K}_\infty, \quad j = 1, \ldots, n$:

- $\psi_{i1}(\|x_i\|_{X_i}) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|_{X_i})$
- $V_i(x_i) \geq \max\left\{ \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U) \right\} \ \Rightarrow \ \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)),$

$\forall x_i \in X_i, \ \forall \tilde{x}_i \in \tilde{X}_i, \forall v \in PC(\mathbb{R}_+, \tilde{X}_i)$ with $v(0) = \tilde{x}_i$.

Small-gain theorem

Gain matrix: $\Gamma_M = (\chi_{ij})_{i,j=1,\ldots,n}$, $\chi_{ij} \in \mathcal{K}_\infty \cup \{0\}$.
Gain operator: $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$

$$\Gamma(s) := \left(\max_{j=1}^{n} \chi_{1j}(s_j), \ldots, \max_{j=1}^{n} \chi_{nj}(s_j) \right), \; s \in \mathbb{R}^n_+.$$

Theorem (Dashkovskiy, M., MCSS, 2013)

Let V_i be ISS-Lyapunov function for Σ_i with gains χ_{ij}.

$$\Gamma(s) \not\geq s, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\} \qquad \text{(SGB)}$$

$\Rightarrow$

- *Σ is ISS.*
- *$V(x) := \max_i\{\sigma_i^{-1}(V_i(x_i))\}$ is a Lyapunov function for Σ.*

For $n = 2$ (SGB) $\quad \Leftrightarrow \quad \chi_{12} \circ \chi_{21} < id$.

Example: Modeling

$$\begin{cases} \frac{\partial s_1}{\partial t} = c_1 \frac{\partial^2 s_1}{\partial x^2} + s_2^2, \quad x \in (0,d), \ t > 0, \\ s_1(0,t) = s_1(d,t) = 0; \\ \frac{\partial s_2}{\partial t} = c_2 \frac{\partial^2 s_2}{\partial x^2} + \sqrt{|s_1|}, \quad x \in (0,d), \ t > 0, \\ s_2(0,t) = s_2(d,t) = 0. \end{cases}$$

- $X_1 = L_2(0,d)$, $U_1 = L_4(0,d)$
- $X_2 = L_4(0,d)$, $U_2 = L_2(0,d)$.
- $X = X_1 \times X_2$.
- $U_{c,i} := C([0,\infty), U_i)$

Define $B_i = c_i \frac{d^2}{dx^2}$,which generate (together with Dirichlet boundary conditions) an analytic semigroup on $L_2(0,d)$ and $L_4(0,d)$ respectively.

Example: Choice of ISS-Lyapunov functions

$$V_1(s_1) = \int_0^d s_1^2(x)dx = \|s_1\|^2_{L_2(0,d)}$$

$$V_2(s_2) = \int_0^d s_2^4(x)dx = \|s_2\|^4_{L_4(0,d)}$$

Consider the Lie derivative of V_1:

$$\begin{aligned}\frac{d}{dt}V_1(s_1) &= 2\int_0^d s_1(x,t)\left(c_1\frac{\partial^2 s_1}{\partial x^2}(x,t) + s_2^2(x,t)\right)dx \\ &\leq -2c_1\left\|\frac{ds_1}{dx}\right\|^2_{L_2(0,d)} + 2\|s_1\|_{L_2(0,d)}\|s_2\|^2_{L_4(0,d)}.\end{aligned}$$

By the Friedrichs' inequality:

$$\begin{aligned}\frac{d}{dt}V_1(s_1) &\leq -2c_1\left(\frac{\pi}{d}\right)^2\|s_1\|^2_{L_2(0,d)} + 2\|s_1\|_{L_2(0,d)}\|s_2\|^2_{L_4(0,d)} \\ &= -2c_1\left(\frac{\pi}{d}\right)^2 V_1(s_1) + 2\sqrt{V_1(s_1)}\sqrt{V_2(s_2)}.\end{aligned}$$

Example: Choice of Lyapunov gains

$$\chi_{12}(r) = \frac{1}{c_1^2 \left(\frac{\pi}{d}\right)^4 (1-\varepsilon_1)^2} r, \quad \forall r > 0 \text{ and any } \varepsilon_1 \in (0,1).$$

We obtain

$$V_1(s_1) \geq \chi_{12}(V_2(s_2)) \quad \Rightarrow \quad \frac{d}{dt} V_1(s_1) \leq -2\varepsilon_1 c_1 \left(\frac{\pi}{d}\right)^2 V_1(s_1).$$

Consider

$$\begin{aligned} \frac{d}{dt} V_2(s_2) &= 4 \int_0^d s_2^3(x,t) \left(c_2 \frac{\partial^2 s_2}{\partial x^2}(x,t) + \sqrt{|s_1(x,t)|} \right) dx \\ &\leq -3c_2 \int_0^d \left(\frac{\partial}{\partial x}(s_2^2) \right)^2 dx + 4 \int_0^d s_2^3(x,t) \sqrt{|s_1(x,t)|} dx \end{aligned}$$

Using Friedrich's and Hölder's inequalities:

$$\frac{d}{dt} V_2(s_2) \leq -3c_2 \left(\frac{\pi}{d}\right)^2 V_2(s_2) + 4(V_2(s_2))^{3/4} (V_1(s_1))^{1/4}.$$

Example: Use of small-gain condition

$$\chi_{21}(r) = \left(\frac{4d^2}{3c_2\pi^2}\right)^4 \frac{1}{(1-\varepsilon_2)^4} r, \ \forall r > 0,$$

where $\varepsilon_2 \in (0,1)$ is an arbitrary constant. It holds the implication

$$V_2(s_2) \geq \chi_{21}(V_1(s_1)) \quad \Rightarrow \quad \frac{d}{dt}V_2(s_2) \leq -\frac{3c_2\pi^2}{d^2}\varepsilon_2 V_2(s_2).$$

Small-gain condition in use:

$$\chi_{12} \circ \chi_{21} < \mathrm{Id} \quad \Leftrightarrow \quad c_1^2 \left(\frac{\pi}{d}\right)^4 (1-\varepsilon_1)^2 \left(\frac{3c_2\pi^2}{4d^2}\right)^4 (1-\varepsilon_2)^4 > 1$$

$$\Leftrightarrow \quad c_1 \left(\frac{\pi}{d}\right)^6 \left(\frac{3c_2}{4}\right)^2 > 1$$

$$\Downarrow$$

0-UGASx

Definition

System Σ is called integral input-to-state stable (iISS) if there exist $\alpha, \mu \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that the inequality

$$\alpha(\|\phi(t, \phi_0, u)\|_X) \leq \beta(\|\phi_0\|_X, t) + \int_0^t \mu(\|u(s)\|_U) ds$$

holds $\forall \phi_0 \in X$, $\forall u \in U_c$ and $\forall t \geq 0$.

Integral input-to-state stability (iISS)

Definition

System Σ is called integral input-to-state stable (iISS) if there exist $\alpha, \mu \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that the inequality

$$\alpha(\|\phi(t,\phi_0,u)\|_X) \leq \beta(\|\phi_0\|_X, t) + \int_0^t \mu(\|u(s)\|_U)ds$$

holds $\forall \phi_0 \in X$, $\forall u \in U_c$ and $\forall t \geq 0$.

Trajectories of iISS systems may be unbounded even if $\sup_{t\geq 0} \|u(t)\|_U < \infty$!

iISS Lyapunov functions

Definition

$V : X \to \mathbb{R}_+$ is an iISS Lyapunov function, if $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$:

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in X$$

and $\forall x \in X, \forall u \in U_c$

$$\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U).$$

Theorem

If $\exists$ iISS-LF for $\Sigma \Rightarrow \Sigma$ is iISS.

Bilinear systems

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + C(x(t), u(t)),\\ x(0) &= \phi_0,\end{aligned} \tag{BL}$$

where $B \in L(U, X)$, $C : X \times U \to X$ and $\exists K > 0$: $\forall x \in X, \forall u \in U$

$$\|C(x,u)\|_X \leq K\|x\|_X\|u\|_U.$$

A bilinear system which is not ISS

$$\begin{aligned}\frac{\partial x}{\partial t} &= \frac{\partial^2 x}{\partial l^2}(l,t) + x(l,t)u(l,t),\ (x,l) \in (0,1) \times (0,\infty)\\ x(0,t) &= x(1,t) = 0.\end{aligned}$$

- $X = C_0[0,1] = \{f \in C[0,1] : f(0) = f(1) = 0\}$
- $U = C[0,1]$

$$Spec(A) = \{-(\pi n)^2 + c|\ n \geq 1\}.$$

Theorem

(BL) *is iISS* $\Leftrightarrow$ (BL) *is 0-UGASx.*

Theorem (Construction of LFs if X is Hilbert)

Let X be a Hilbert space and A generate an analytic semigroup on X, and $\exists$ coercive positive self-adjoint operator $P \in L(X)$ satisfying

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle = -\|x\|_X^2, \quad \forall x \in D(A)$$

Then iISS-LF of (BL) *can be chosen as:*

$$W(x) = \ln\Big(1 + \langle Px, x\rangle\Big).$$

Example

$$
\begin{aligned}
\frac{\partial x}{\partial t}(l,t) &= c\frac{\partial^2 x}{\partial l^2}(l,t) + \frac{x(l,t)}{1+|l-1|x(l,t)^2}u(l,t),\\
x(0,t) &= x(L,t) = 0;
\end{aligned}
$$

Proposition

System is ISS for $X = L_2(0,L)$ and $U = L_2(0,L)$ and all $L < 1$.

Proposition

System is iISS for $X = L_2(0,L)$ and $U = C(0,L)$ and all $L > 0$.

Example

Define

$$W(x) = \int_0^L x^2(l)dl = \|x\|^2_{L_2(0,L)}.$$

$$\dot{W}(x) \leq -2c\left(\frac{\pi}{L}\right)^2 W(x) + 2W(x)\|u\|_{C(0,L)}$$

Choosing

$$V(x) = \ln\left(1 + W(x)\right)$$

yields

$$\begin{aligned}\dot{V}(x) &\leq -2c\Big(\frac{\pi}{L}\Big)^2 \frac{W(x)}{1+W(x)} + 2\frac{W(x)}{1+W(x)}\|u\|_{C(0,L)} \\ &\leq \underbrace{-2c\Big(\frac{\pi}{L}\Big)^2 (1-e^{-V(x)})}_{\alpha(V(x))} + \underbrace{2\|u\|_{C(0,L)}}_{\sigma(\|u\|_U)}.\end{aligned}$$

Summary and Outlook

Construction of ISS-Lyapunov functions → Existence of ISS-Lyapunov function → ISS

Discussed results

- Lyapunov sufficient conditions for ISS and integral ISS
- ISS and iISS of linear and bilinear systems
- Small-gain theorems for interconnections of ISS systems

Related results: impulsive systems

- Dwell-time conditions for ISS of impulsive systems
- Small-gain theorems for impulsive systems

Papers on which a talk is based

- S. Dashkovskiy, A.M. Input-to-state stability of infinite-dimensional control systems, MCSS, 2013.
- A.M., Hiroshi Ito. Integral input-to-state stability of bilinear systems, in preparation, 2014
- S. Dashkovskiy, A.M. Input-to-State Stability of Nonlinear Impulsive Systems, SICON, 2013.

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Thank you for attention!