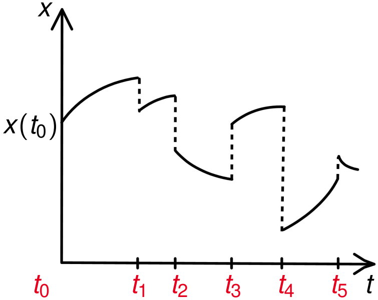
Dwell-time conditions for input-to-state stability of impulsive systems

Andrii Mironchenko

Centre for Industrial Mathematics, University of Bremen, Germany

March 09, 2012
Stuttgart, Germany

Impulsive systems



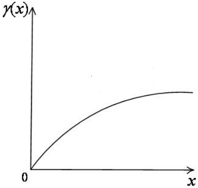
$$\dot{x}(t) = f(x(t), u(t)) \quad , \; t \notin \{t_1, t_2, \dots\},$$
$$x(t) = g(x^-(t), u^-(t)) \quad , \; t \in \{t_1, t_2, \dots\}.$$

$u \in L_{\infty,\, loc}([t_0, \infty), \mathbb{R}^m)$, $x \in \mathbb{R}^n$ is abs. continuous between impulses,
$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is loc. Lip. continuous.

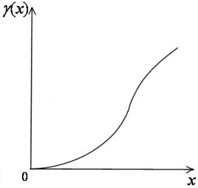
$x^-(t) := \lim_{s \nearrow t} x(s)$, $u^-(t) := \lim_{s \nearrow t} u(s)$.

Comparison functions

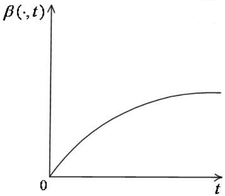
$\gamma \in \mathcal{K}$

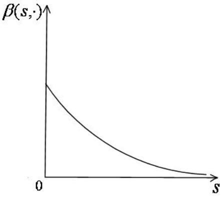


$\gamma \in \mathcal{K}_\infty$



$\beta \in \mathcal{KL}$



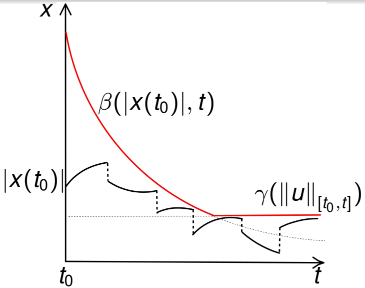


Definition (Input-to-state stability (ISS))

Σ is **ISS** for a given impulse time sequence T, if $\exists \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, s.t.

$$|x(t)| \leq \max\{\beta(|x(t_0)|, t - t_0), \gamma(\|u\|_{[t_0,t]})\}$$

holds for all $x(t_0) \in \mathbb{R}^n$, $u \in L_{\infty,\ loc}([t_0,\infty),\mathbb{R}^m)$, $t \geq t_0$.
Σ is **uniformly ISS** w.r.t. a class S of impulse time sequences, if it is ISS $\forall T \in S$, and the functions β and γ do not depend on the choice of $T \in S$.



Exponential ISS-Lyapunov functions

$$\Sigma : \begin{array}{l} \dot{x}(t) = f(x(t), u(t)),\ t \neq t_k, \\ x(t) = g(x^-(t), u^-(t)),\ t = t_k,\ k \in \mathbb{N}. \end{array}$$

Definition

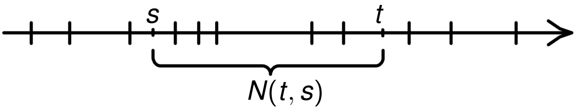
A Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an **exponential ISS-Lyapunov function** with rate coefficients c, d for Σ if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), x \in \mathbb{R}^n$$

holds and $\exists c, d \in \mathbb{R}$ and $\gamma \in \mathcal{K}_\infty$ such that

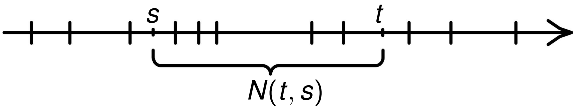
$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -cV(x) \\ V(g(x,u)) \leq e^{-d} V(x). \end{cases}$$

Average dwell time (ADT) condition



$N(t, s)$ is the number of impulse times t_k in $(s, t]$.

Average dwell time (ADT) condition



$N(t,s)$ is the number of impulse times t_k in $(s,t]$.

Theorem (Hespanha, Liberzon, Teel 2008)

Let V be an exponential ISS-LF for Σ with $c, d \in \mathbb{R}$, $d \neq 0$. For arbitrary $\mu, \lambda > 0$, let $\mathcal{S}[\mu,\lambda]$ denote the class of impulse time sequences $\{t_k\}$ satisfying

ADT: $-dN(t,s) - (c-\lambda)(t-s) \leq \mu, \quad$ *for any* $s,t : 0 \leq s \leq t.$

Then Σ is uniformly ISS over $\mathcal{S}[\mu,\lambda]$.

Generalized ADT condition

Theorem

Let V be an exponential ISS-Lyapunov function for Σ with corresponding coefficients $c \in \mathbb{R}$, $d \neq 0$.
For arbitrary $h : \mathbb{R}_+ \to (0,\infty)$, for which there exist $g \in \mathcal{L}$: $h(x) \leq g(x)$ for all $x \in \mathbb{R}_+$ consider the class $\mathcal{S}[h]$ of impulse time-sequences, satisfying gADT condition:

$$-dN(t,s) - c(t-s) \leq \ln h(t-s), \quad \forall t \geq s \geq t_0.$$

Then Σ is uniformly ISS over $\mathcal{S}[h]$.

Corollary

Taking $h(x) = e^{\mu - \lambda x}$ for all $x \in \mathbb{R}_+$, we obtain the ADT condition.

Tightness of gADT condition

Consider

$$\dot{x} = -cx,$$
$$x(t) = e^{-d} x^{-}(t)$$

with initial condition $x(0) = x_0$. Its solution for arbitrary time sequence T is given by

$$x(t) = e^{-dN(t,t_0)-c(t-t_0)} x_0.$$

If T does not satisfy the gADT condition, then $e^{-dN(t,t_0)-c(t-t_0)}$ cannot be estimated from above by $\mathcal{L}$-function, and consequently, the system under consideration is not GAS.

ISS-Lyapunov functions (ISS-LF)

$$\Sigma: \quad \begin{aligned} \dot{x}(t) &= f(x(t), u(t)),\ t \neq t_k, \\ x(t) &= g(x^-(t), u^-(t)),\ t = t_k,\ k \in \mathbb{N}. \end{aligned}$$

Definition (ISS-Lyapunov function)

A Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an **ISS-Lyapunov function** for Σ if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), x \in \mathbb{R}^n$$

holds and $\exists \gamma \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$, such that

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x)), & \text{f. a. a. } x, u \\ V(g(x,u)) \leq \alpha(V(x)), \ \forall x, u \end{cases}$$

ISS via nonexponential ISS-Lyapunov functions

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x)) \\ V(g(x,u)) \leq \alpha(V(x)). \end{cases}$$

Define $S_\theta := \{\{t_i\}_1^\infty \subset [t_0, \infty) \ : \ t_{i+1} - t_i \geq \theta, \ \forall i \in \mathbb{N}\}$.

Theorem

Let V be an ISS-Lyapunov function for Σ with $\varphi, \alpha \in \mathcal{P}$. Let for some $\theta, \delta > 0$ and all $a > 0$ it holds

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta. \qquad (1)$$

Then Σ is ISS for all impulse time sequences $T \in S_\theta$.

ISS via nonexponential ISS-Lyapunov functions-2

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x)) \\ V(g(x,u)) \leq \alpha(V(x)). \end{cases}$$

Define $\tilde{S}_\theta := \{\{t_i\}_1^\infty \subset [t_0, \infty) \;:\; t_{i+1} - t_i \leq \theta, \; \forall i \in \mathbb{N}\}$.

Theorem

Let V be an ISS-Lyapunov function for Σ, α is as in the Definition of ISS-LF and $-\varphi \in \mathcal{P}$. Let for some $\theta, \delta > 0$ and all $a > 0$ it holds

$$\int_{\alpha(a)}^{a} \frac{ds}{\varphi(s)} \geq \theta + \delta. \qquad (2)$$

Then Σ is ISS w.r.t. every sequence from $\tilde{S}_\theta$.

General theorem for exponential ISS-LFs

Let V be an exponential ISS Lyapunov function with $d < 0$:

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -cV(x) \\ V(g(x,u)) \leq e^{-d} V(x). \end{cases}$$

That is $\varphi(s) = cs$ and $\alpha(s) = e^{-d}s$ for some $c, d \in \mathbb{R}$.

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} = \frac{-d}{c} \leq \theta - \delta \Leftrightarrow \frac{1}{\theta} \leq \frac{c-\lambda}{-d}, \text{ for some } \lambda > 0.$$

For the given $\lambda > 0$ the smallest θ (which corresponds to the largest S_θ) is given by $\theta_* = \frac{-d}{c-\lambda}$.

Relation between fixed and average dwell-time conditions

$$\text{ADT: } -dN(t,s)-(c-\lambda)(t-s)\leq \mu, \quad \text{for any } s,t: 0\leq s\leq t.$$

Lemma

Let $c>0$ be given and let $d<0$. Then it holds $S_{\theta_*}=\mathcal{S}[-d,\lambda]$.

If $\mu < -d$, then

$$N(t,s)-\frac{c-\lambda}{-d}(t-s)\leq \frac{\mu}{-d}<1.$$

For small enough $t-s$ we have $N(t,s)<1$, i.e. $N(t,s)=0$.
Thus, if $\mu<-d$, the impulses are not allowed.

Summary

What do we need to verify ISS of the impulsive systems?

Summary

What do we need to verify ISS of the impulsive systems?

- ISS-Lyapunov functions
- Restrictions on the set of impulse time sequences

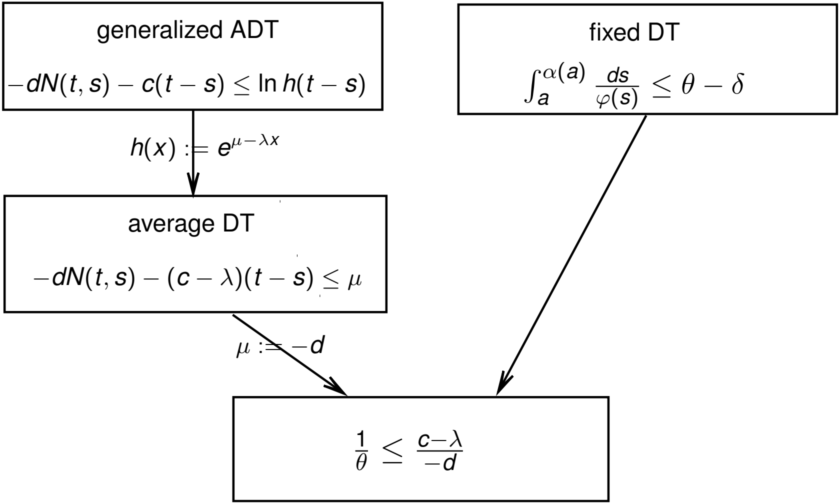
Summary

What do we need to verify ISS of the impulsive systems?

- ISS-Lyapunov functions
- Restrictions on the set of impulse time sequences

What kind of restrictions do we need?

Summary


generalized ADT
$-dN(t,s) - c(t-s) \leq \ln h(t-s)$
fixed DT
$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta$
$h(x) := e^{\mu - \lambda x}$
average DT
$-dN(t,s) - (c-\lambda)(t-s) \leq \mu$
$\mu := -d$
$\frac{1}{\theta} \leq \frac{c-\lambda}{-d}$

Possible generalizations

Let X be state space, U - space of input values, and let both be Banach.

$$\begin{aligned}\dot{x}(t) &= Ax(t) + f(x(t), u(t)) \quad &, \ t \notin \{t_1, t_2, \ldots\},\\ x(t) &= g(x^-(t), u^-(t)) \quad &, \ t \in \{t_1, t_2, \ldots\}.\end{aligned}$$

Here $x(t) \in X$, $u(t) \in U$ and A is a generator of C_0-semigroup over X.

f is loc. Lip. continuous.

$x^-(t) := \lim_{s \nearrow t} x(s)$, $u^-(t) := \lim_{s \nearrow t} u(s)$.

Possible directions of future work

- Methods for construction of ISS-Lyapunov functions.
- ISS of impulsive differential equations with time-delays.
- Generalization of results to the hybrid systems.

Supported by the German Research Foundation (DFG) as a part of Collaborative Research Centre 637 "Autonomous Cooperating Logistic Processes - A Paradigm Shift and its Limitations".

SFB 637
Autonomous Logistics

Thank you for attention!