On characterizations of Input-to-State Stability for Infinite-Dimensional Systems

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Class of systems

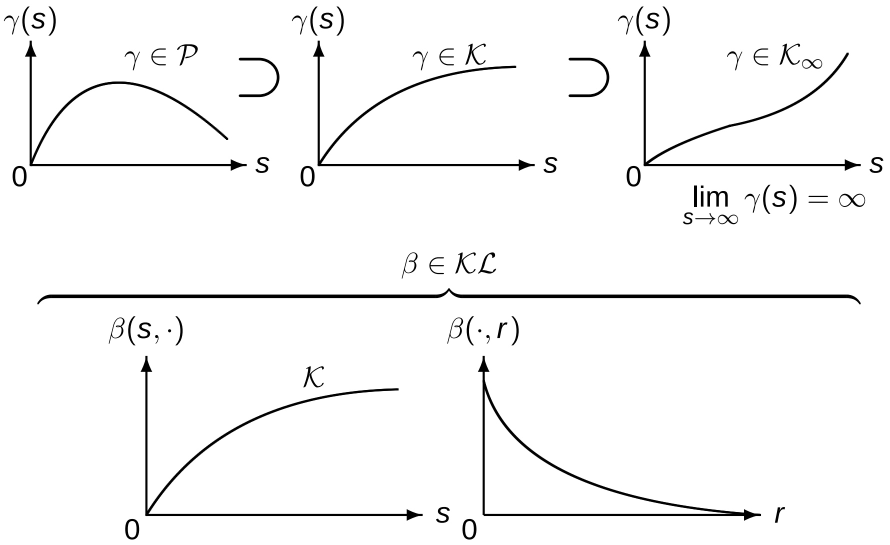
$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X, \\ x(0) = \phi_0. \end{cases}$$

- X = Banach space
- $\mathcal{U} = PC(\mathbb{R}_+, U)$
- A generates C_0-semigroup T

$x \in C([0,T],X)$ is a mild solution iff

$$x(t) = T(t)\phi_0 + \int_0^t T(t-s)f(x(s),u(s))ds.$$

Comparison functions



Input-to-state stability

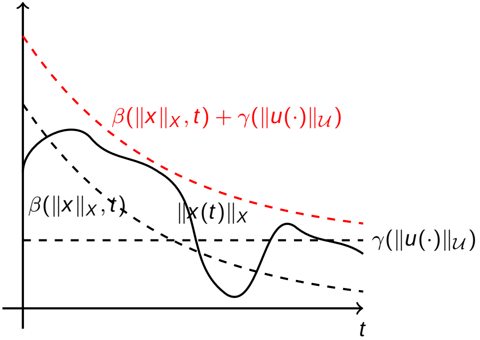
Definition (ISS)

ISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty: \quad \forall t, x, u$

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u(\cdot)\|_{\mathcal{U}}). \qquad (1)$$

LISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty, r > 0:$

(1) holds for $x : \|x\|_X \leq r$, $u : \|u\|_{\mathcal{U}} \leq r$, $\forall t$.



Definition (GAS uniform w.r.t. state (0-UGASs))

0-UGASs $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}: \quad \forall t, x$

$$\|\phi(t,x,0)\|_X \leq \beta(\|x\|_X, t).$$

Definition (Asymptotic Gain)

AG $:\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty: \quad \forall x, u$

$$\limsup_{t\to\infty} \|\phi(t,x,u)\|_X \leq \gamma(\|u(\cdot)\|_{\mathcal{U}}).$$

Definition (Global stability)

GS $:\Leftrightarrow \quad \exists \sigma, \gamma \in \mathcal{K}_\infty: \quad \forall t, x, u$

$$\|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u(\cdot)\|_{\mathcal{U}}).$$

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),$$
$$x(0) = \phi_0,$$

Definition

$V : X \to \mathbb{R}_+$ is ISS-Lyapunov function iff $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$, $\sigma, \alpha \in \mathcal{K}$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $\dot{V}_u(x) \leq -\alpha(V(x)) + \sigma(\|u(0)\|_U)$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x)).$$

Theorem

$\exists$ *(L)ISS Lyapunov function* $\Rightarrow$ *(L)ISS*

Motivation

$$\dot{x} = f(x, u) \qquad \text{(ODE)}$$

Theorem (Characterizations of ISS (Sontag, Wang))

For (ODE) *the following statements are equivalent:*

1. *ISS*
2. $\exists$ *ISS-LF*
3. *AG+ GS*
4. *AG + 0-UGASs*
5. *AG + LISS*
6. ...

- E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. Sys. & Cont. Letters, 1995.
- E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. IEEE TAC, 1996.

Just a lemma

$$\dot{x} = f(x, u) \qquad \text{(ODE)}$$

Lemma (Sontag, Wang, IEEE TAC, 1996)

For (ODE) *if*

- *Let f be Lipschitz w.r.t. x uniformly w.r.t. u.*
- *Let $f(x, \cdot)$ be continuous for all x.*

Then

$$\text{0-UGASs} \Rightarrow \text{LISS}$$

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$$\text{0-UGASs} \Rightarrow \text{LISS}$$

Does
0-UGASs $\Rightarrow$ LISS
hold in ∞-dimensions?

Example

$$\dot{x}_k(t) = -\frac{1}{1 + k|u(t)|} x_k(t)$$

- $X = l_1 := \{(x_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |x_k| < \infty\}$
- $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$

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- $u \equiv 0 \quad \Rightarrow \quad \|\phi(t,x,0)\|_X \leq e^{-t}\|x\|_X \quad \Rightarrow$ 0-UGASs
- $\forall u \in \mathcal{U} \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \leq \|x\|_X \quad \Rightarrow$ GS with zero gain
- AG with zero gain

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But it is not LISS!

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0-UGASs + AG $\not\Rightarrow$ LISS

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Can we fix it?

$$\dot{x}(t) = Ax(t) + f(x(t), u(t))$$

Theorem (Converse LISS Lyapunov theorem)

- $\forall C > 0\ \exists K(C) > 0$: $\|x\|_X \le C$ and $\|y\|_X \le C \quad \Rightarrow$
$$\|f(y,v) - f(x,v)\|_X \le L_f(C)\|y - x\|_X.$$
- $f(x,\cdot)$ is continuous for all $x \in X$.
- $\exists\ \sigma \in \mathcal{K}$ and $\rho > 0$: $\|v\|_U \le \rho$ and $\|x\|_X \le \rho \quad \Rightarrow$
$$\|f(x,v) - f(x,0)\|_X \le \sigma(\|v\|_U).$$

$$\Downarrow$$

0-UASs $\Leftrightarrow$ ∃ 0-UAS LF $\Leftrightarrow$ ∃ LISS-LF $\Leftrightarrow$ LISS

Example: reload

$$\dot{x}_k(t) = -\frac{1}{1+|u(t)|^k} x_k(t)$$

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- $\forall u \in \mathcal{U} \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \leq \|x\|_X \quad \Rightarrow$ GS with zero gain
- $\|u\|_{\mathcal{U}} \leq 1 \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \leq e^{-\frac{t}{2}}\|x\|_X \quad \Rightarrow$ LISS
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- AG with zero gain

But it is not ISS!

AG + GS + 0-UGASs + LISS $\not\Rightarrow$ ISS

Summary and Outlook

We discussed

- Uniformity matters!
- Certain uniformity of rhs $\Rightarrow$ 0-UAS = LISS = $\exists$ LISS-LF
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We could discuss

- AG + GS and AG + 0-UGASs are different notions
- AG + LS $\not\Rightarrow$ AG + GS
- Characterizations of ISS
- A notion of weak ISS and its characterizations

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- Certain uniformity of rhs $\Rightarrow$ 0-UAS = LISS = $\exists$ LISS-LF
- AG + GS + 0-UGASs + LISS $\not\Rightarrow$ ISS

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