Input-to-state stability of infinite-dimensional systems: recent results and open problems.

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joint work with

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Seminar
Würzburg
10 February 2017

Outline

Class of systems

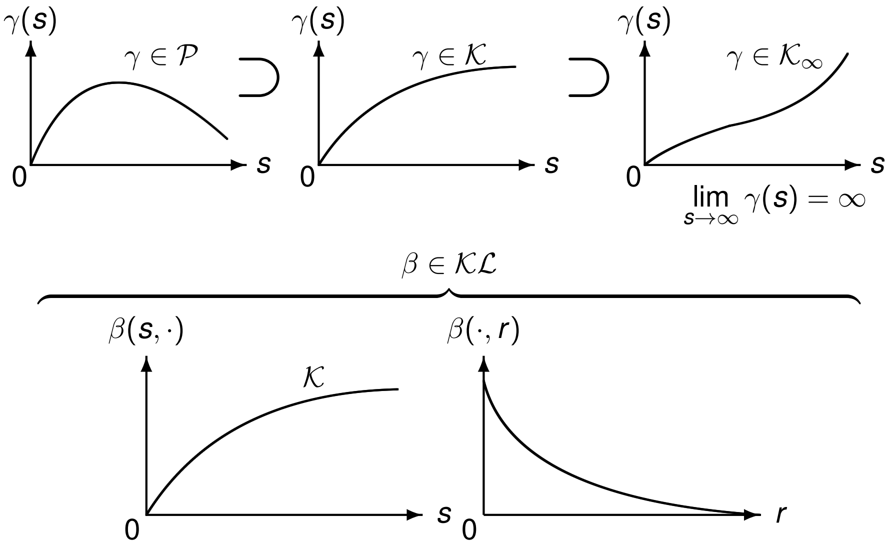
$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), & x(t) \in D(A) \subset X, \\ x(0) = x_0. \end{cases}$$

- $\mathcal{U} = PC(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.
- T is a C_0-semigroup.
- f is a Lipschitz continuous perturbation.

$x \in C([0, T], X)$ is a mild solution iff

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s), u(s))ds.$$

Comparison functions

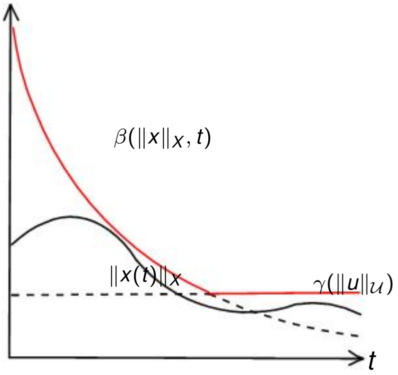


Input-to-state stability

Definition (ISS)

ISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall x \in X, \forall u \in \mathcal{U}$

$$\|\phi(t,x,u)\|_X \leq \max\Big\{\beta(\|x\|_X, t), \underbrace{\gamma}_{Gain}\Big(\sup_{s\in[0,t]} \|u(s)\|_U\Big)\Big\}.$$



Motivation

Why ISS?

1. Unified theory of internal and external stability
   - E. D. Sontag. *Input to State Stability: Basic Concepts and Results.* In Nonlinear and Optimal Control Theory, chapter 3, 2008.
2. Robust stabilization of nonlinear systems
   - M. Krstić, I. Kanellakopoulos, P. Kokotović. Nonlinear and adaptive control design, Wiley, 1995.
3. Design of robust nonlinear observers
   - M. Arcak, P. Kokotović. Nonlinear observers: a circle criterion design and robustness analysis, 2001.
4. Stability of networks of nonlinear control systems
   - Z.-P. Jiang, I. Mareels, Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems, Automatica, 1996.
   - S. Dashkovskiy, B. Rüffer, F. Wirth. Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions, SICON, 2010.
5. ...

Objectives

1. Constructions of Lyapunov functions for interconnected PDEs
2. Lyapunov characterization of ISS and LISS
3. Non-Lyapunov characterizations
   - ISS and its characterizations via novel uniform limit property
   - Strong ISS and its characterization
   - UGAS characterization
   - Overview of ISS Characterizations
4. Differences between ISS theory for ODEs and for ∞-dim systems.
5. Literature overview
6. Open problems

LISS and 0-UGAS

Definition (GAS uniform w.r.t. state (0-UGAS))

0-UGAS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}: \quad \forall x \in X, \forall t \geq 0$

$$\|\phi(t,x,0)\|_X \leq \beta(\|x\|_X, t).$$

Definition (LISS)

LISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty, r > 0$:

$t \geq 0, \ \|x\|_X \leq r, \ \|u\|_{\mathcal{U}} \leq r \quad \Rightarrow$

$$\|\phi(t,x,u)\|_X \leq \max\Big\{\beta(\|x\|_X, t), \underbrace{\gamma}_{Gain}\Big(\sup_{s\in[0,t]} \|u(s)\|_U\Big)\Big\}.$$

Outline

ISS Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),$$

Definition

$V : X \to \mathbb{R}_+$ is a non-coercive ISS-Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \neq 0$

(ii) $\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \quad \forall x \in X, \forall u \in \mathcal{U},$

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}\big(V(\phi(t,x,u)) - V(x)\big).$$

V is called a coercive ISS-Lyapunov function if

$$\exists \psi_1, \psi_2 \in \mathcal{K}_\infty : \quad \psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \neq 0.$$

Theorem (Direct Lyapunov theorem)

$\exists$ *a coercive ISS Lyapunov function* $\quad \Rightarrow \quad$ *ISS.*

Example of nonexponentially ISS system

$$\begin{cases} \frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial l^2} + ax - x\left(\frac{\partial x}{\partial l}\right)^2 + u, & \forall t > 0, \\ x(0,t) = x(\pi,t) = 0. \end{cases}$$

Different choices of input space are possible!

Proposition

Let $X = H_0^1(0,\pi)$.

1. If $U = W_0^{1,\frac{4}{3}}(0,\pi) \Rightarrow$ ISS iff $a \leq 1$.
2. If $U = L_2(0,\pi) \Rightarrow$ ISS provided $a < 1$.

$$V(x) = \int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl = \|x\|^2_{H_0^1(0,\pi)}.$$

The derivative of V along the trajectories is equal

$$\begin{aligned}\dot{V}(x) \quad = \quad & \underbrace{-2\int_0^\pi \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl}_{Linear\ \ dynamics} + 2a\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl \\ & \underbrace{-\frac{2}{3}\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^4 dl}_{Nonlinear\ \ dynamics} - 2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl.\end{aligned}$$

$$V(x) = \int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl = \|x\|^2_{H^1_0(0,\pi)}.$$

The derivative of V along the trajectories is equal

$$\dot V(x) = \underbrace{-2\int_0^\pi \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl}_{Linear\ \ dynamics} + 2a\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl$$

$$\underbrace{-\frac{2}{3}\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^4 dl}_{Nonlinear\ \ dynamics} - 2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl.$$

$$U = L_2(0,\pi) \ \Rightarrow \ -2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl \le 2\omega \int_0^\pi \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl + 2\frac{1}{\omega}\int_0^\pi u^2 dl$$

$$V(x) = \int_0^{\pi} \left(\frac{\partial x}{\partial l}\right)^2 dl = \|x\|^2_{H_0^1(0,\pi)}.$$

The derivative of V along the trajectories is equal

$$\dot V(x) = \underbrace{-2\int_0^{\pi} \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl}_{Linear\ \ dynamics} + 2a \int_0^{\pi} \left(\frac{\partial x}{\partial l}\right)^2 dl$$

$$\underbrace{-\frac{2}{3}\int_0^{\pi} \left(\frac{\partial x}{\partial l}\right)^4 dl}_{Nonlinear\ \ dynamics} - 2\int_0^{\pi} \frac{\partial^2 x}{\partial l^2} u dl.$$

$$U = L_2(0,\pi) \ \Rightarrow \ -2\int_0^{\pi} \frac{\partial^2 x}{\partial l^2} u dl \leq 2\omega \int_0^{\pi} \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl + 2\frac{1}{\omega}\int_0^{\pi} u^2 dl,$$

$$U = W_0^{1,\frac{4}{3}}(0,\pi) \ \Rightarrow \ -2\int_0^{\pi} \frac{\partial^2 x}{\partial l^2} u dl \leq \frac{\omega}{4}\int_0^{\pi} \left|\frac{\partial x}{\partial l}\right|^4 dl + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{4}\int_0^{\pi} \left|\frac{\partial u}{\partial l}\right|^{\frac{4}{3}} dl$$

Case $U = L_2(0, \pi)$

$$\dot{V}(x) \leq -2(1 - a - \omega)V(x) - \frac{2}{3\pi}V^2(x) + 2\frac{1}{\omega}\int_0^\pi u^2 dl.$$

- $a < 1 \Rightarrow$ ISS for $U = L_2(0, \pi)$.
- $a = 1 \Rightarrow$ 0-UGAS for $U = L_2(0, \pi)$.
- $a = 1 \Rightarrow$ Is it ISS?

Case $U = L_2(0,\pi)$

$$\dot{V}(x) \leq -2(1-a-\omega)V(x) - \frac{2}{3\pi}V^2(x) + 2\frac{1}{\omega}\int_0^\pi u^2 dl.$$

- $a < 1 \Rightarrow$ ISS for $U = L_2(0,\pi)$.
- $a = 1 \Rightarrow$ 0-UGAS for $U = L_2(0,\pi)$.
- $a = 1 \Rightarrow$ Is it ISS?

Case $U = W_0^{1,\frac{4}{3}}(0,\pi)$

$$\dot{V}(x) \leq -2(1-a)V(x) - \Big(\frac{1}{3} - \frac{\omega}{2}\Big)\frac{1}{\pi}V^2(x) + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{2}\|u\|^{\frac{4}{3}}_{W_0^{1,\frac{4}{3}}(0,\pi)}.$$

- $a \leq 1 \Rightarrow$ ISS for $U = W_0^{1,\frac{4}{3}}(0,\pi)$

Case $U = L_2(0,\pi)$

$$\dot{V}(x) \leq -2(1-a-\omega)V(x) - \frac{2}{3\pi}V^2(x) + 2\frac{1}{\omega}\int_0^\pi u^2 dl.$$

- $a < 1 \Rightarrow$ ISS for $U = L_2(0,\pi)$.
- $a = 1 \Rightarrow$ 0-UGAS for $U = L_2(0,\pi)$.
- $a = 1 \Rightarrow$ Is it ISS?

Case $U = W_0^{1,\frac{4}{3}}(0,\pi)$

$$\dot{V}(x) \leq -2(1-a)V(x) - \Big(\frac{1}{3} - \frac{\omega}{2}\Big)\frac{1}{\pi}V^2(x) + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{2}\|u\|^{\frac{4}{3}}_{W_0^{1,\frac{4}{3}}(0,\pi)}.$$

- $a \leq 1 \Rightarrow$ ISS for $U = W_0^{1,\frac{4}{3}}(0,\pi)$

Σ is no more 0-UGAS for $a > 1$.

Definition (GAS uniform w.r.t. state (0-UGASs))

0-UGASs $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}: \quad \forall x \in X, \forall t \geq 0$

$$\|\phi(t,x,0)\|_X \leq \beta(\|x\|_X, t).$$

Definition (ISS)

ISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall x \in X, \forall u \in U_c$

$$\|\phi(t,x,u)\|_X \leq \max\Big\{\beta(\|x\|_X, t), \underbrace{\gamma}_{Gain}\Big(\sup_{s\in[0,t]} \|u(s)\|_U\Big)\Big\}.$$

Definition (integral input-to-state stability (iISS))

iISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \alpha, \mu \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall x \in X, \forall u \in U_c$

$$\alpha(\|\phi(t,x,u)\|_X) \leq \beta(\|x\|_X, t) + \int_0^t \mu(\|u(s)\|_U)ds.$$

Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),$$

Definition

$V : X \to \mathbb{R}_+$ is an iISS-Lyapunov function iff $\exists \psi_1, \psi_2, \sigma, \alpha \in \mathcal{K}$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $\dot{V}_u(x) \leq -\alpha(V(x)) + \sigma(\|u(0)\|_U)$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x)).$$

Theorem

$\exists$ *ISS/iISS Lyapunov function* $\Rightarrow$ *ISS/iISS.*

Interconnections of two integral ISS systems

$$\Sigma : \begin{cases} \Sigma_1 : & \dot{x}_1 = A_1 x_1 + f_1(x_1, x_2, u), \ x_1 \in X_1 \\ \Sigma_2 : & \dot{x}_2 = A_2 x_2 + f_2(x_1, x_2, u), \ x_2 \in X_2 \end{cases}$$

ISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is iISS-Lyapunov functions for Σ_i, $i = 1, 2$ iff

- $\dot{V}_1(x_1) \leq -\alpha_1(\|x_1\|_{X_1}) + \sigma_1(\|x_2\|_{X_2}) + \kappa_1(\|u(0)\|_U)$,
- $\dot{V}_2(x_2) \leq -\alpha_2(\|x_2\|_{X_2}) + \sigma_2(\|x_1\|_{X_1}) + \kappa_2(\|u(0)\|_U)$,

Small-gain theorems: Overview

Small-gain theorems for ODEs

1. Jiang, Teel, Praly. Small-gain theorem for ISS systems and applications, MCSS, 1994.
2. Jiang, Mareels, Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems, Automatica, 1996.
3. Dashkovskiy, Rüffer, Wirth. An ISS small gain theorem for general networks, MCSS, 2007.
4. Dashkovskiy, Rüffer, Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions, SICON, 2010.
5. Dashkovskiy, Ito, Wirth. On a small gain theorem for ISS networks in dissipative Lyapunov form, EJC, 2011.

Small-gain theorems for ∞-dim systems

1. Dashkovskiy, M. Input-to-state stability of infinite-dimensional control systems, MCSS, 2013.
2. M., Ito. Construction of Lyapunov functions for interconnected parabolic systems: an iISS approach, SICON, 2015.

Small-gain theorem for 2 interconnected iISS systems

Theorem (A. Mironchenko, H. Ito)

Let:

- $\lim_{s\to\infty} \alpha_i(s) = \infty$ *or* $\lim_{s\to\infty} \sigma_{3-i}(s)\kappa_i(1) < \infty$ *for i=1,2.*
- $\exists c > 1$*:* $\forall s \in \mathbb{R}_+$*:* $\psi_{11}^{-1} \circ \psi_{12} \circ \alpha_1^{\ominus} \circ c\sigma_1 \circ \psi_{21}^{-1} \circ \psi_{22} \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \leq s.$

$\Rightarrow$ *Σ is iISS.*

If additionally

- $\alpha_i \in \mathcal{K}_\infty$ *for* $i = 1, 2$ $\Rightarrow$ *Σ is ISS.*

iISS (ISS) Lyapunov function:

$$V(x) = \int_0^{V_1(x_1)} \lambda_1(s)ds + \int_0^{V_2(x_2)} \lambda_2(s)ds.$$

$V_1(x_1) = \psi_1(\|x_1\|_X)$, $V_2(x_2) = \psi_2(\|x_2\|_X)$ $\Rightarrow$ SGC is

$$\exists c > 1\colon \forall s \in \mathbb{R}_+\colon \alpha_1^{\ominus} \circ c\sigma_1 \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \leq s.$$

Example

$$
\begin{cases}
\dfrac{\partial x_1}{\partial t}(l,t) = \dfrac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\
x_1(0,t) = x_1(\pi,t) = 0; \\
\dfrac{\partial x_2}{\partial t} = \dfrac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2\Big(\dfrac{\partial x_2}{\partial l}\Big)^2 + \Big(\frac{x_1^2}{1+x_1^2}\Big)^{\frac{1}{2}}, \\
x_2(0,t) = x_2(\pi,t) = 0.
\end{cases}
$$

Example

$$\begin{cases} \dfrac{\partial x_1}{\partial t}(l,t) = \dfrac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\ x_1(0,t) = x_1(\pi,t) = 0; \\ \dfrac{\partial x_2}{\partial t} = \dfrac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2\Big(\dfrac{\partial x_2}{\partial l}\Big)^2 + \Big(\frac{x_1^2}{1+x_1^2}\Big)^{\frac{1}{2}}; \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases}$$

For what a, b is this system UGASs?

Example

$$
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\frac{\partial x_1}{\partial t}(l,t) = \frac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\
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\end{cases}
$$

For what a, b is this system UGASs?

State spaces: $X_1 := L_2(0,\pi)$ and $X_2 := H_0^1(0,\pi)$

Strategy

1. x_1-subsystem is iISS
2. x_2-subsystem is ISS
3. Interconnection is UGASs

x_1-subsystem is iISS

$$\begin{cases} \frac{\partial x_1}{\partial t}(l,t) = \frac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\ x_1(0,t) = x_1(\pi,t) = 0; \end{cases}$$

iISS-Lyapunov function:

$$V_1(x_1) := \ln\left(1 + \|x_1\|^2_{L_2(0,\pi)}\right)$$

Lie derivative of V_1:

$$\dot{V}_1 \leq -\frac{2\|x_1\|^2_{L_2(0,\pi)}}{1 + \|x_1\|^2_{L_2(0,\pi)}} + 2\|x_2\|^4_{L_\infty(0,\pi)}.$$

Finally, via Agmon's inequality:

$$\dot{V}_1(x_1) \leq -\frac{2\|x_1\|^2_{L_2(0,\pi)}}{1 + \|x_1\|^2_{L_2(0,\pi)}} + 8\|x_2\|^4_{H^1_0(0,\pi)}.$$

$$\begin{cases} \dfrac{\partial x_2}{\partial t} = \dfrac{\partial^2 x_2}{\partial l^2} + a x_2 - b x_2 \Big(\dfrac{\partial x_2}{\partial l}\Big)^2 + \overbrace{\Big(\dfrac{x_1^2}{1+x_1^2}\Big)^{\frac{1}{2}}}^{u}, \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases}$$

ISS-Lyapunov function:

$$V_2(x_2) = \int_0^\pi \Big(\frac{\partial x_2}{\partial l}\Big)^2 dl = \|x_2\|^2_{H_0^1(0,\pi)}.$$

Lie derivative of V_2:

$$\dot{V}_2(x_2) \leq -2(1-a-\omega)V_2(x_2) - \frac{2b}{3\pi}V_2^2(x_2) + 2\frac{1}{\omega}\int_0^\pi u^2 dl.$$

$$\dot{V}_2 \leq -2(1-a-\frac{\omega}{2})\|x_2\|^2_{H_0^1(0,\pi)} - \frac{2b}{3\pi}\|x_2\|^4_{H_0^1(0,\pi)} + \frac{\pi}{\omega}\left(\frac{\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}\right).$$

Interconnection is UGASs

Condition for UGASs: for some $c > 0$, for all $s \in \mathbb{R}_+$

$$\psi_{11}^{-1} \circ \psi_{12} \circ \alpha_1^{\ominus} \circ c\sigma_1 \circ \psi_{21}^{-1} \circ \psi_{22} \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \leq s$$

In our case: $\psi_{11}(s) = \psi_{12}(s) = \ln(1+s^2), \quad \psi_{21}(s) = \psi_{22}(s) = s^2$

$$\alpha_1(s) = \frac{2s^2}{1+s^2}, \quad \sigma_1(s) = 8s^4, \quad \kappa_1(s) = 0$$

$$\alpha_2(s) = 2\left(1 - a - \frac{\omega}{2}\right)s^2 + \frac{2b}{3\pi}s^4, \ \sigma_2(s) = \frac{\pi}{\omega}\left(\frac{s^2}{1+s^2}\right), \ \kappa_2(s) = 0$$

Condition for UGASs

$$a + \frac{3\pi^2}{b} < 1, \quad b \geq 0.$$

What Next?

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),$$

Now several questions arise:

1. ISS ??? $\Rightarrow$??? $\exists$ a coercive ISS Lyapunov function
   For ODEs positively answered in
   - E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. Sys. & Cont. Letters, 1995.

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2. What about local ISS?

What Next?

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2. What about local ISS?
3. What about other characterizations of ISS?

Outline

LISS = 0-UAS = ∃ coercive LISS-LF

$$\dot{x}(t) = Ax(t) + f(x(t), u(t))$$

Theorem (AM, Sys. & Cont. Lett., 2016)

(i) $\forall C > 0\ \exists K(C) > 0$:

$$\|x\|_X \leq C,\ \|y\|_X \leq C \quad \Rightarrow \quad \|f(y,v) - f(x,v)\|_X \leq L_f(C)\|y - x\|_X.$$

(ii) *$f(x,\cdot)$ be continuous for all $x \in X$.*

(iii) $\exists\ \sigma \in \mathcal{K}$ *and* $\rho > 0$:

$$\|v\|_U \leq \rho,\ \|x\|_X \leq \rho \quad \Rightarrow \quad \|f(x,v) - f(x,0)\|_X \leq \sigma(\|v\|_U).$$

$$\Downarrow$$

0-UAS $\Leftrightarrow$ ∃ *0-UAS LF* $\Leftrightarrow$ ∃ *LISS LF* $\Leftrightarrow$ *LISS*

Counterexample: 0-UGAS but not LISS

$$\dot{x}_k(t) = -\frac{1}{1 + k|u(t)|} x_k(t)$$

- $X = l_1 := \{(x_k)_{k=1}^\infty : \sum_{k=1}^\infty |x_k| < \infty\}$
- $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$

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- $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$

- $u \equiv 0 \quad \Rightarrow \quad \|\phi(t, x, 0)\|_X \leq e^{-t} \|x\|_X$

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- $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$

- $u \equiv 0 \quad \Rightarrow \quad \|\phi(t, x, 0)\|_X \leq e^{-t} \|x\|_X$
- $\forall u \in \mathcal{U} \quad \Rightarrow \quad \|\phi(t, x, u)\|_X \leq \|x\|_X$

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- $u \equiv 0 \quad \Rightarrow \quad \|\phi(t, x, 0)\|_X \leq e^{-t}\|x\|_X$
- $\forall u \in \mathcal{U} \quad \Rightarrow \quad \|\phi(t, x, u)\|_X \leq \|x\|_X$
- $\forall u, \ \forall x \quad \Rightarrow \quad \|\phi(t, x, u)\|_X \to 0, t \to \infty$

Counterexample: 0-UGAS but not LISS

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- $X = l_1 := \{(x_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |x_k| < \infty\}$
- $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$

- $u \equiv 0 \quad \Rightarrow \quad \|\phi(t,x,0)\|_X \leq e^{-t}\|x\|_X$
- $\forall u \in \mathcal{U} \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \leq \|x\|_X$
- $\forall u, \ \forall x \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \to 0, t \to \infty$
- Properties (i) and (ii) hold

Counterexample: 0-UGAS but not LISS

$$\dot{x}_k(t) = -\frac{1}{1+k|u(t)|}x_k(t)$$

- $X = l_1 := \{(x_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |x_k| < \infty\}$
- $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$

- $u \equiv 0 \quad \Rightarrow \quad \|\phi(t,x,0)\|_X \leq e^{-t}\|x\|_X$
- $\forall u \in \mathcal{U} \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \leq \|x\|_X$
- $\forall u,\ \forall x \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \to 0, t \to \infty$
- Properties (i) and (ii) hold

But property (iii) does not hold!

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- Properties (i) and (ii) hold

But property (iii) does not hold!

It is not LISS!

Last remarks concerning LISS

$\mathbb{R}^n$-world

1. In $\mathbb{R}^n$ (ii) $\wedge$ (i) $\quad\Rightarrow\quad$ (iii).

Last remarks concerning LISS

$\mathbb{R}^n$-world

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2. Sontag and Wang, 1996: 0-GAS $\Rightarrow$ LISS for ODEs.

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4. Our result is more general and uses other technique.

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2. Sontag and Wang, 1996: $\quad$ 0-GAS $\quad\Rightarrow\quad$ LISS for ODEs.
3. An adaptation of the argument by Sontag and Wang would give 0-AS = LISS.
4. Our result is more general and uses other technique.

Open question

1. If (iii) is dropped, will converse LISS Lyapunov theorem still hold, even though 0-UAS $\neq$ LISS in general?

Converse Lyapunov theorems for global ISS

Converse Lyapunov theorems for the global ISS property are more complicated!

Our plan

1. UGAS = $\exists$ UGAS Lyapunov function
2. ISS = UGAS under suitable feedback

ISS and Robust Stability

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X \qquad (\Sigma)$$

- Let $u(t) := d(t)\varphi(x(t))$.
- with $d \in \mathcal{D} = \{d : \mathbb{R}_+ \to D\}, \quad D = \{d \in U : \|d\|_U \leq 1\}$.

Then

$$\dot{x}(t) = Ax(t) + \underbrace{f(x(t), d(t)\varphi(x(t)))}_{g(x(t),d(t))}. \qquad (\tilde{\Sigma})$$

Definition

Σ is weakly uniformly robustly asymptotically stable (WURS), if $\exists$ Lipschitz continuous $\varphi : X \to \mathbb{R}_+$ and $\psi \in \mathcal{K}_\infty$:

- $\varphi(x) \geq \psi(\|x\|_X)$
- $\tilde{\Sigma}$ is UGAS over $\mathcal{D} = \{d : \mathbb{R}_+ \to D\}$.

Converse Lyapunov Theorem for disturbed systems

Lemma

f is bi-Lipschitz on bounded balls $\Rightarrow$ g is Lipschitz continuous on bounded balls, uniformly w.r.t. the second argument.

Theorem (Karafyllis, Jiang, 2011)

$$\dot{x}(t) = Ax(t) + g(x(t), d(t)). \qquad (\tilde{\Sigma})$$

- *g is Lipschitz continuous on bounded balls, uniformly w.r.t. the second argument*
- *$\tilde{\Sigma}$ is UGAS*

$\Rightarrow \exists$ *Lipschitz continuous Lyapunov function for $\tilde{\Sigma}$.*

Proposition

V is a Lipschitz continuous Lyapunov function for $\tilde{\Sigma}$
$\Rightarrow$ V is a Lipschitz continuous ISS Lyapunov function for Σ.

ISS and Robust Stability

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X \qquad (\Sigma)$$

$$\dot{x}(t) = Ax(t) + \underbrace{f(x(t), d(t)\varphi(x(t)))}_{g(x(t),d(t))}. \qquad (\tilde{\Sigma})$$

Theorem

Σ *is ISS* $\Rightarrow$ Σ *is WURS.*

$$\begin{array}{ccc}
\Sigma \text{ is ISS} & \Longleftarrow & \exists \text{ ISS-LF for } \Sigma \\
\Downarrow & & \Uparrow \\
\tilde{\Sigma} \text{ is UGAS} & \Longrightarrow & \exists \text{ UGAS-LF for } \tilde{\Sigma}
\end{array}$$

Intermediate conclusion

Constructions of Lyapunov Functions

- Lyapunov theory for parabolic PDEs with polynomial nonlinearities
- iISS Lyapunov small-gain theorems, for nonlinear PDE systems with in-domain couplings.
- Choice of state space matters!

Lyapunov Characterizations

- Certain uniformity of rhs $\Rightarrow$ 0-UAS $\Leftrightarrow$ LISS $\Leftrightarrow$ $\exists$ LISS-LF
- AG $\wedge$ GS $\wedge$ 0-UGASs $\wedge$ LISS $\not\Rightarrow$ ISS
- ISS $\Leftrightarrow$ $\exists$ ISS-LF
- Uniformity matters!

Intermediate conclusion

Papers:

- A.M., H. Ito. *Construction of Lyapunov functions for interconnected parabolic systems: an iISS approach*, SICON, 2015.
- A.M. *Local input-to-state stability: Characterizations and counterexamples.* Sys. & Con. Lett., 2016.
- A.M., F. Wirth. *Lyapunov characterization of input-to-state stability for control systems over Banach spaces.* To be submitted to SCL, 2016.

Papers and slides can be found at
www.mironchenko.com

Outline

Literature overview: linear systems

$$\dot{x} = Ax + Bu$$

- Systems with unbounded B are studied
- PDEs with boundary control fall into this class
- This theory is just in the beginning of the development

- F. B. Argomedo, E. Witrant, C. Prieur. *D1-Input-to-state stability of a time-varying nonhomogeneous diffusive equation subject to boundary disturbances*, Proc. of ACC, 2012.
- B. Jacob, R. Nabiullin, J. R. Partington, F. Schwenninger. *Infinite-dimensional input-to-state stability and Orlicz spaces*, Submitted.
- I. Karafyllis, M. Krstic. *ISS with respect to boundary disturbances for 1-D parabolic PDEs*, IEEE TAC, 2016.
- I. Karafyllis, M. Krstic. *ISS in Different Norms for 1-D Parabolic PDES With Boundary Disturbances*, Submitted.
- A. M., I. Karafyllis, M. Krstic. *Monotonicity Methods for ISS of Nonlinear Parabolic PDEs with Boundary Disturbances*, work in progress.

Literature overview: nonlinear systems 1

Lur'e, monotone, hyperbolic and impulsive systems

- B. Jayawardhana, H. Logemann, E. P. Ryan, *Infinite-Dimensional Feedback Systems: the Circle Criterion and Input-to-State Stability*, Comm. Inf. Syst. 2008.
- S. Dashkovskiy, A.M. On the uniform input-to-state stability of reaction-diffusion systems, CDC 2010.
- C. Prieur, F. Mazenc. ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws, MCSS, 2012.
- S. Dashkovskiy, A. M. *Input-to-state stability of nonlinear impulsive systems*, SICON, 2013.

Literature overview: nonlinear systems 2

Constructions of Lyapunov functions. Small-gain theorems

- F. Mazenc and C. Prieur, *Strict Lyapunov functions for semilinear parabolic partial differential equations*, MCRF, 2011.
- I. Karafyllis, Z.-P. Jiang, *A vector small-gain theorem for general non-linear control systems*, IMA Journal, 2011.
- S. Dashkovskiy, A. M. *Input-to-state stability of infinite-dimensional control systems*, MCSS, 2013.
- A.M., H. Ito. *Construction of Lyapunov functions for interconnected parabolic systems: an iISS approach*, SICON, 2015.
- A.M., H. Ito, *Characterizations of integral input-to-state stability for bilinear systems in infinite dimensions*, MCRF, 2016.
- M. Ahmadi, G. Valmorbida, A. Papachristodoulou, *Dissipation inequalities for the analysis of a class of PDEs*, Automatica, 2016.

Literature overview: nonlinear systems 3

Constructions of Lyapunov functions. Small-gain theorems

- A. Chaillet, G. Is. Detorakis, S. Palfi, S. Senova. Robust stabilization of delayed neural fields with partial measurement and actuation, submitted, 2016.
- A. Pisano, Y. Orlov. On the ISS properties of a class of parabolic DPS with discontinuous control using sampled-in-space sensing and actuation, submitted.
- A. Tanwani, C. Prieur, S. Tarbouriech. Input-to-State Stabilization in H^1-Norm for Boundary Controlled Linear Hyperbolic PDEs with Application to Quantized Control, CDC 2016.

Characterizations of ISS

- A.M. Local input-to-state stability: Characterizations and counterexamples. Sys. & Con. Lett., 2016.
- A.M., F. Wirth. Non-coercive Lyapunov functions for infinite-dimensional systems. Submitted to Transactions of AMS, 2016.
- A.M., F. Wirth. *Lyapunov characterization of input-to-state stability for control systems over Banach spaces.* To be submitted to SCL, 2017.
- A.M., F. Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. To be submitted to TAC, 2017

Outline

In this part of the talk we characterize UGAS property:

- How to decompose UGAS into simpler properties?
- Which combinations of properties are equivalent to UGAS?
- Is there is something new in infinite-dimensional theory?

What properties should possess an UGAS system?

First of all, the solutions should exist for all times!

Definition

Σ is called forward complete (FC) if for any $x \in X$, any $u \in \mathcal{U}$ the solution $\phi(\cdot, x, u)$ exist and is finite for all times.

Thinking about stability theory

Does forward completeness tell us something beyond mere existence of solutions?

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Definition

FC Σ has bounded reachability sets (BRS) $:=$ $\forall R > 0\ \forall \tau > 0$ $\exists M = M(R,\tau)$:

$$\sup_{\|x\|_X \le R,\ \|u\|_{\mathcal{U}} \le R,\ t \in [0,\tau]} \|\phi(t,x,u)\|_X \le M(R,\tau) < \infty.$$

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Proposition (Lin, Sontag, Wang, 1996)

$$\Sigma_{ODE} : \dot{x} = f(x,u), \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m$$

If Σ_{ODE} is FC $\quad \Rightarrow \quad \Sigma_{ODE}$ is BRS.

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BRS is a "bridge" between solution theory and stability theory

Thinking about stability theory

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Definition

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If Σ_{ODE} is FC $\Rightarrow$ Σ_{ODE} is BRS.

Infinite-dimensional FC systems are not necessarily BRS!

BRS gives us bounds on finite time intervals. But we are interested in global in time bounds!

Definition (Stability-like notions)

0-UGS $:\Leftrightarrow \quad \exists \sigma \in \mathcal{K}_\infty : \forall t \geq 0,\ \forall x \in X$

$$\|\phi(t,x,0)\|_X \leq \sigma(\|x\|_X) \tag{1}$$

0-ULS $:\Leftrightarrow \quad$ (1) holds $\forall x : \|x\|_X \leq r$.

0-pUGS $:\Leftrightarrow \quad \exists \sigma \in \mathcal{K}_\infty,\ c > 0 : \forall t \geq 0,\ \forall x \in X$

$$\|\phi(t,x,0)\|_X \leq \sigma(\|x\|_X) + c$$

Definition (Attractivity for zero inputs)

0-GATT $:\Leftrightarrow \quad \forall x \in X \quad \Rightarrow \quad \lim_{t\to\infty} \|\phi(t,x,0)\|_X = 0.$

0-GAS $:\Leftrightarrow \quad$ 0-ULS $\wedge$ 0-GATT.

Linear undisturbed systems

Consider systems

$$\dot{x} = Ax$$

the solutions are:

$$\phi(t, x) = T(t)x$$

Simple properties

- BRS: $\exists M, \lambda > 0 \quad \Rightarrow \quad \|T(t)x\|_X \leq Me^{\lambda t}\|x\|_X$.
- 0-GATT $\Rightarrow$ 0-GS
  - $\forall x \in X \quad \Rightarrow \quad \sup_{t \geq 0} \|T(t)x\|_X < \infty$.
  - Banach-Steinhaus Theorem: $\sup_{t \geq 0} \|T(t)\| = K < \infty$.
  - $\|T(t)x\|_X \leq K\|x\|_X$.
- 0-GAS $\not\Rightarrow$ 0-UGAS
  - $X = c_0 := \{x = \{x_k\}_{k=1}^{\infty} : \lim_{k \to \infty} x_k = 0\}$
  - $\dot{x}_k = -\frac{1}{k}x_k, \quad k \in \mathbb{N}$

Linear undisturbed systems-2

$$\|T(t)x\|_X \leq Me^{wt}\|x\|_X,\ w < 0 \Longleftrightarrow \text{0-UGAS}$$

$$\Downarrow \quad \not\Uparrow$$

$$\|T(t)x\|_X \to 0, \quad \forall x \in X \Longleftrightarrow \text{0-GAS}$$

$$\Downarrow \quad \not\Uparrow$$

$$\|T(t)\| \leq M \text{ for some } M > 0$$

$$\Downarrow \quad \not\Uparrow$$

$$\|T(t)x\|_X \leq Me^{wt}\|x\|_X,\ w \in \mathbb{R} \Longleftrightarrow \text{BRS} \Longleftrightarrow \text{FC}$$

Figure : Relations between stability notions for linear systems

Nonlinear undisturbed systems: examples

Consider systems

$$\dot{x} = Ax + f(x)$$

- FC $\wedge$ 0-UAS $\wedge$ 0-GAS $\not\Rightarrow$ BRS

$$X = l_2 := \{(z_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |z_i|^2 < \infty\}, \quad z_i = (x_i, y_i) \in \mathbb{R}^2.$$

$$\Sigma : \begin{cases} \Sigma_i : \begin{cases} \dot{x}_i = -x_i + x_i^2 y_i - \frac{1}{i^2} x_i^3, \\ \dot{y}_i = -y_i. \end{cases} \\ i = 1, \ldots, \infty \end{cases}$$

Nonlinear undisturbed systems: examples

Consider systems

$$\dot{x} = Ax + f(x)$$

- FC $\wedge$ 0-UAS $\wedge$ 0-GAS $\not\Rightarrow$ BRS

$$X = l_2 := \{(z_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |z_i|^2 < \infty\}, \quad z_i = (x_i, y_i) \in \mathbb{R}^2.$$

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- FC $\wedge$ BRS $\wedge$ 0-UAS $\wedge$ 0-GAS $\not\Rightarrow$ 0-UGS

- Pick previous system.
- Change (nonuniformly) time clocks: $\tilde{x}_k(t) := x_k(\frac{t}{k})$
- Resulting system is 0-GAS, BRS, but not 0-UGS.

Moral

For ∞-dim systems global attractivity
$\not\Rightarrow$ uniform boundedness on solutions

Moral

For ∞-dim systems global attractivity $\not\Rightarrow$ uniform boundedness on solutions

ODEs	∞-dim
closed unit ball is compact	closed unit ball is never compact
FC $\Rightarrow$ BRS	FC $\wedge$ 0-UAS $\wedge$ 0-GAS $\not\Rightarrow$ BRS
0-GATT $\Rightarrow$ 0-pUGS	FC $\wedge$ BRS $\wedge$ 0-UAS $\wedge$ 0-GAS $\not\Rightarrow$ 0-UGS
0-GAS $\Leftrightarrow$ 0-UGAS	?????

Moral

For ∞-dim systems global attractivity
$\not\Rightarrow$ uniform boundedness on solutions

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Generalizing results from ODE theory is not straightforward!

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Non-Lyapunov characterizations due to Sontag, Wang
do not hold for in ∞-dim!

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Generalizing results from ODE theory is not straightforward!

Non-Lyapunov characterizations due to Sontag, Wang
do not hold for in ∞-dim!

Key: Uniform weak attractivity!

Definition (Attractivity for zero inputs)

$$\text{0-GATT} \;:\Leftrightarrow\; \forall x \in X \quad \Rightarrow \quad \lim_{t\to\infty} \|\phi(t,x,0)\|_X = 0.$$

$$\text{0-LIM} \;:\Leftrightarrow\; \forall \varepsilon > 0,\ x \in X\ \exists \tau = \tau(\varepsilon, x):$$

$$\|\phi(t,x,0)\|_X \le \varepsilon.$$

$$\text{0-UGATT} \;:\Leftrightarrow\; \forall \varepsilon, \delta > 0\ \exists T = T(\varepsilon,\delta):$$

$$t \ge T,\ \|x\|_X \le \delta \quad \Rightarrow \quad \|\phi(t,x,0)\|_X \le \varepsilon.$$

$$\text{0-ULIM} \;:\Leftrightarrow\; \forall \varepsilon, \delta > 0\ \exists \tau = \tau(\varepsilon,\delta):$$

$$\|x\|_X \le \delta \quad \Rightarrow \quad \exists t \le \tau : \|\phi(t,x,0)\|_X \le \varepsilon.$$

Weak attractivity

Origins of the limit property

- 0-LIM is also called weak attractivity and was introduced in N. P. Bhatia. Weak attractors in dynamical systems, Bol. Sot. Mat. Mex., 1966
- Weak attractivity has been extensively studied in the book N. P. Bhatia, G- P. Szegö. Stability theory of dynamical systems, 1967.
- Sontag, Wang have extended this notion to systems with inputs and called it LIM.
- The notion of uniform weak attractivity (0-ULIM) is due to A.M., F. Wirth. Non-coercive Lyapunov functions for infinite-dimensional systems, submitted, 2016.

Proposition

$$\dot{x} = f(x)$$

For ODEs weak attractivity is equivalent to uniform weak attractivity

Follows from technical results in Sontag, Wang, 1996.

Undisturbed infinite-dimensional systems

Results for ODEs

- FC $\Rightarrow$ BRS
- 0-LIM $\Leftrightarrow$ 0-ULIM
- 0-LIM $\Rightarrow$ 0-pUGS
- 0-UGATT $\Leftrightarrow$ 0-UGAS
-

Results for ∞-dim systems

- FC $\wedge$ 0-UAS $\wedge$ 0-GAS $\not\Rightarrow$ BRS
- 0-GATT $\wedge$ 0-UGS $\not\Rightarrow$ 0-ULIM
- 0-ULIM $\wedge$ BRS $\Rightarrow$ 0-pUGS
- 0-UGATT $\wedge$ BRS $\Leftrightarrow$ 0-UGAS
- 0-UGATT $\Leftrightarrow$ 0-ULIM $\wedge$ 0-ULS

Undisturbed infinite-dimensional systems

Let Σ be BRS. Then:

$$\text{0-ULIM} \wedge \text{0-ULS} \Longleftrightarrow \text{0-UGATT} \Longleftrightarrow \text{0-UGAS} \Longleftrightarrow \exists \text{ LF}$$

$$\text{(1)} \Downarrow \qquad\qquad \text{(1)(2)} \Downarrow$$

$$\text{0-LIM} \wedge \text{0-UGS} \Longleftrightarrow \text{0-GAS} \wedge \text{0-UGS} \quad \begin{matrix} \not\Rightarrow \\ \not\Leftarrow \end{matrix} \quad \text{0-UAS} \wedge \text{0-GATT}$$

$$\text{(1)(2)} \Downarrow \qquad\qquad \text{(1)} \Downarrow$$

$$\text{0-LIM} \wedge \text{0-ULS} \Longleftrightarrow \text{0-GAS} \Longleftrightarrow \text{0-GATT} \wedge \text{0-ULS}$$

Implications become equivalences for

1. ODE systems
2. Linear systems

Undisturbed infinite-dimensional systems

Let Σ be BRS. Then:

$$
\begin{array}{ccccccc}
\text{0-ULIM} \wedge \text{0-ULS} & \Longleftrightarrow & \text{0-UGATT} & \Longleftrightarrow & \text{0-UGAS} & \Longleftrightarrow & \exists\ \text{LF} \\
 & & (1) \Downarrow & & & & (1)(2) \Downarrow \\
\text{0-LIM} \wedge \text{0-UGS} & \Longleftrightarrow & \text{0-GAS} \wedge \text{0-UGS} & & \begin{array}{c} \not\Rightarrow \\ \not\Leftarrow \end{array} & & \text{0-UAS} \wedge \text{0-GATT} \\
 & & (1)(2) \Downarrow & & & & (1) \Downarrow \\
\text{0-LIM} \wedge \text{0-ULS} & \Longleftrightarrow & \text{0-GAS} & & \Longleftrightarrow & & \text{0-GATT} \wedge \text{0-ULS}
\end{array}
$$

For ODEs: 0-LIM $\Leftrightarrow$ 0-ULIM, FC $\Rightarrow$ BRS

Outline

Decompositions of ISS

1. For ODEs:
   - Asymptotic stability = stability + attractivity.
2. ISS implies:
   - $t \geq 0, x \in X \quad \Rightarrow \quad \|\phi(t,x,0)\|_X \leq \beta(\|x\|_X, t)$.
   - $x \in X \quad \Rightarrow \quad \limsup_{t\to\infty} \|\phi(t,x,u)\|_X \leq \gamma(\|u\|_{\mathcal{U}})$.
3. Are above two properties taken together equivalent to ISS?

Decompositions of ISS are important since:

- They give a better understanding of ISS.
- Are an important tool for proof of other results in ISS theory, e.g. small-gain theorems.

Definition (Stability-like notions)

UGS $:\Leftrightarrow \quad \exists \sigma, \gamma \in \mathcal{K}_\infty$:

$$t \geq 0, x \in X, u \in \mathcal{U} \;\Rightarrow\; \|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}).$$

pUGS $:\Leftrightarrow \quad \exists \sigma, \gamma \in \mathcal{K}_\infty, c > 0$:

$$t \geq 0, x \in X, u \in \mathcal{U} \;\Rightarrow\; \|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}) + c.$$

ULS $:\Leftrightarrow \quad \exists \sigma, \gamma \in \mathcal{K}_\infty, r > 0$:

$$t \geq 0, \|x\|_X \leq r, \|u\|_{\mathcal{U}} \leq r \;\Rightarrow\; \|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}).$$

Lemma

UGS $\quad\Leftrightarrow\quad$ pUGS $\wedge$ ULS

Definition (Attractivity-like notions)

$$\text{AG} :\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty \cup \{0\}: \forall x \in X,\ \forall u \in \mathcal{U}$$

$$\limsup_{t\to+\infty} \|\phi(t,x,u)\|_X \le \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{AG} :\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty: \forall x \in X, \forall u \in \mathcal{U}, \forall \varepsilon > 0\ \exists T = T(\varepsilon, x, u):$$

$$t \ge T \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{sAG} :\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty: \forall x \in X, \forall \varepsilon > 0\ \exists T = T(\varepsilon, x):$$

$$t \ge T,\ u \in \mathcal{U} \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{UAG} :\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty \cup \{0\}: \forall \varepsilon, \delta > 0\ \exists T = T(\varepsilon, \delta):$$

$$t \ge T,\ u \in \mathcal{U},\ \|x\|_X \le \delta \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

Definition (LIM notions)

$$\text{LIM} \;:\Leftrightarrow\; \exists\gamma \in \mathcal{K}_\infty \cup \{0\}\colon \forall x \in X,\ \forall u \in \mathcal{U}$$
$$\inf_{t\geq 0} \|\phi(t,x,u)\|_X \leq \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{LIM} \;:\Leftrightarrow\; \exists\gamma \in \mathcal{K}_\infty\colon \forall x \in X, \forall u \in \mathcal{U}, \forall\varepsilon > 0\ \exists T = T(\varepsilon,x,u):$$
$$\|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{sLIM} \;:\Leftrightarrow\; \exists\gamma \in \mathcal{K}_\infty\colon \forall x \in X, \forall\varepsilon > 0\ \exists T = T(\varepsilon,x):$$
$$u \in \mathcal{U} \quad\Rightarrow\quad \exists t \leq T : \|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{ULIM} \;:\Leftrightarrow\; \exists\gamma \in \mathcal{K}_\infty \cup \{0\}\colon \forall\varepsilon,\delta > 0\ \exists T = T(\varepsilon,\delta):$$
$$u \in \mathcal{U},\ \|x\|_X \leq \delta \quad\Rightarrow\quad \exists t \leq T : \|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

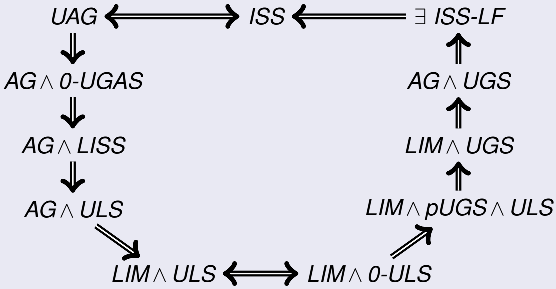
- AG, UAG, LIM notion were introduced by Sontag, Wang (IEEE, 1996).
- sAG, sLIM, ULIM notion were introduced by M., Wirth (2016).

Motivation

$$\dot{x} = f(x, u) \qquad \text{(ODE)}$$

Theorem (Characterizations of ISS (Sontag, Wang))

For (ODE) *the following statements are equivalent:*



- E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. Sys. & Cont. Letters, 1995.
- E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. IEEE TAC, 1996.

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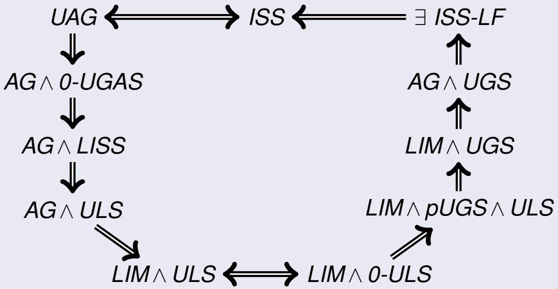

What does remain from this picture in ∞ dimensions?

Theorem (AM, F. Wirth, 2016)

Let Σ be BRS. Then the following statements are equivalent:

(i) Σ *is ISS*

(ii) Σ *is UAG*

(iii) Σ *is ULIM* $\wedge$ *ULS*

(iv) Σ *is ULIM* $\wedge$ *UGS*

If in addition $\exists\ \sigma \in \mathcal{K}$ *and* $\rho > 0$*:*

$$\|v\|_U \leq \rho,\ \|x\|_X \leq \rho \quad \Rightarrow \quad \|f(x,v) - f(x,0)\|_X \leq \sigma(\|v\|_U).$$

then the following notion is also equivalent to ISS:

(v) Σ *is ULIM* $\wedge$ *0-ULS*

Proof.

Lemmas, which show our result:

- ISS $\Rightarrow$ UAG
- UAG $\Rightarrow$ ULIM (clear); UAG $\Rightarrow$ ULS
- ULIM $\wedge$ BRS $\Rightarrow$ pUGS
- pUGS $\wedge$ ULS $\Leftrightarrow$ UGS
- ULIM $\wedge$ BRS $\wedge$ ULS $\Rightarrow$ pUGS $\wedge$ ULS $\Leftrightarrow$ UGS
- ULIM $\wedge$ UGS $\Rightarrow$ UAG
- UAG $\wedge$ UGS $\Rightarrow$ ISS
- Under extra property we have 0-UAS $\Leftrightarrow$ LISS
- 0-ULIM $\wedge$ 0-ULS $\Rightarrow$ 0-UGAS $\Rightarrow$ ULS

$\square$

Characterizations of strong ISS

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X \qquad (\Sigma)$$

Definition

Σ is strongly input-to-state stable (sISS), if $\exists \gamma \in \mathcal{K}$, $\sigma \in \mathcal{K}_\infty$ and $\beta : X \times \mathbb{R}_+ \to \mathbb{R}_+$:

1. $\beta(x, \cdot) \in \mathcal{L}$ for all $x \in X$, $x \neq 0$
2. $\beta(x, t) \leq \sigma(\|x\|_X)$ for all $x \in X$ and all $t \geq 0$
3. for all $x \in X$, all $u \in \mathcal{U}$ and all $t \geq 0$ it holds that

$$\|\phi(t, x, u)\|_X \leq \beta(x, t) + \gamma(\|u\|_\mathcal{U}).$$

Theorem

$$sISS \quad \Leftrightarrow \quad sAG \wedge UGS \quad \Leftrightarrow \quad sLIM \wedge UGS$$

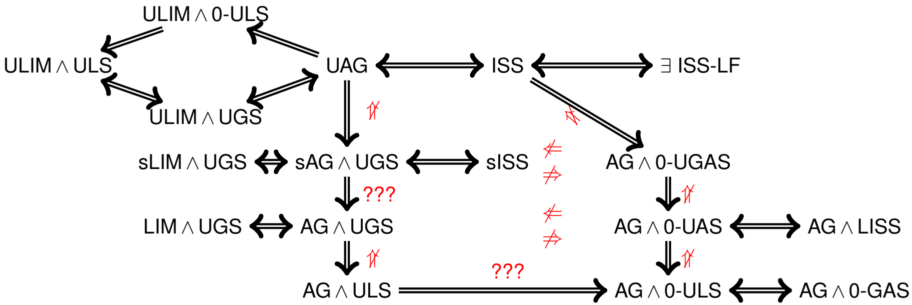

Figure : Relations for BRS systems

Corollary

For ODEs all these notions are equivalent, since:

- *FC* $\Leftrightarrow$ *BRS*
- *LIM* $\Leftrightarrow$ *ULIM*

Summary

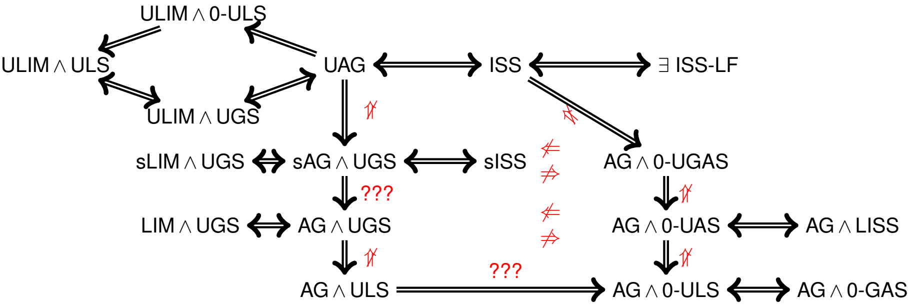

Figure : Relations for BRS systems

- A.M. *Local input-to-state stability: Characterizations and counterexamples.* Sys. & Con. Lett., 2016
- A.M., F. Wirth. *Characterizations of input-to-state stability for in?nite-dimensional systems.* Submitted to TAC, 2017
- A.M., F. Wirth. *Global converse Lyapunov theorems for infinite-dimensional systems.* Proc. of the NOLCOS 2016

Summary

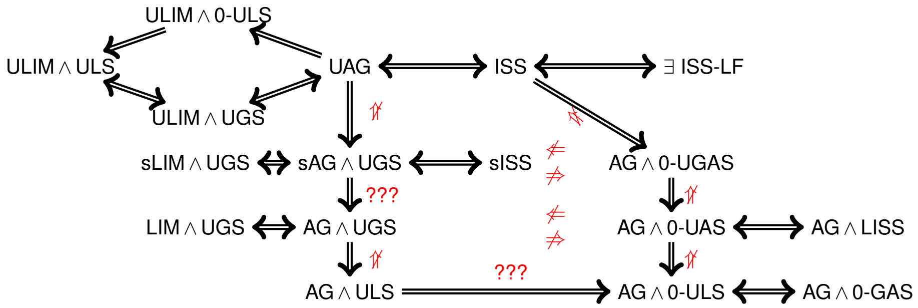

Figure : Relations for BRS systems

What we discussed

- Converse Lyapunov theorem for LISS
- Converse Lyapunov theorem for ISS
- Nonuniform notions $\not\Rightarrow$ Uniform notions
- There are several nonequivalent groups of notions.

Summary

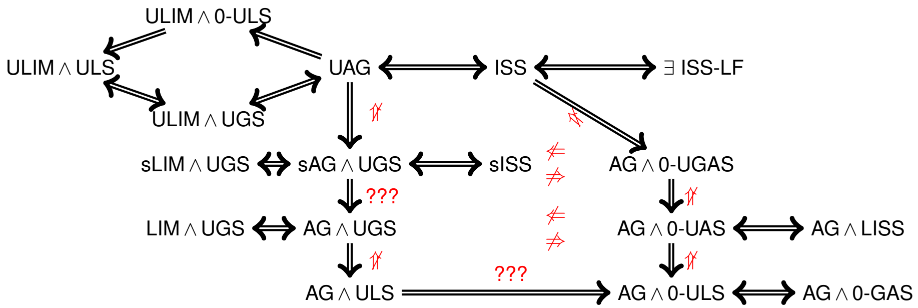

Figure : Relations for BRS systems

What we could discuss

- Generalizations of these results to wide classes of systems, including:
  - DEs in Banach spaces
  - Time-delay systems
  - Switched systems

Outline

Motivation

- There is a stability theory for dynamical systems over locally compact metric spaces:

  N. P. Bhatia, G- P. Szegö. Stability theory of dynamical systems, 1967.

- $\mathbb{R}^n$ is locally compact.
- ∞-dim normed linear spaces are never locally compact.
- There is a hope that many results of Sontag, Wang may hold for control systems over locally compact metric spaces.

Aim

- Develop ISS characterizations for control systems over metric spaces
- Specialize it to locally compact metric spaces

0.2 strong ISS, strong LIM, strong AG

Motivation

- We have only basic results concerning characterizations of AG.
- Relations between LIM and AG are not clear, even for ODEs
- Relations between sLIM and LIM are again not clear, even for ODEs
- The same about sAG and AG.

Aim

- Obtain deep characterizations of strong ISS property.

Motivation

- For linear systems with admissible operators iISS $\Rightarrow$ ISS.
- But it is unclear, whether ISS $\Rightarrow$ iISS.
- The same about nonlinear systems.

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- For linear systems with admissible operators iISS $\Rightarrow$ ISS.
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- The same about nonlinear systems.

Aim

- Characterizations of integral ISS property are needed!

0.3 Characterizations of integral ISS

Motivation

- For linear systems with admissible operators iISS $\Rightarrow$ ISS.
- But it is unclear, whether ISS $\Rightarrow$ iISS.
- The same about nonlinear systems.

Open problems on "Characterizations"

- Characterizations in metric spaces
- Characterizations of sISS, sLIM, sAG notions
- Characterizations of integral ISS

$$\dot{x}(t) = Ax(t) + f(x(t)),$$

$V : X \to \mathbb{R}_+$ is a non-coercive Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \neq 0$

(ii) $\dot{V}(x) \leq -\alpha(\|x\|_X) \quad \forall x \in X,$

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Theorem (AM, F. Wirth, NOLCOS 2016)

Let:

- *Σ be RFC*
- *0 be a robust equilibrium of Σ*

If V is a noncoercive Lyapunov function for Σ $\Rightarrow$ Σ is UGAS

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Let:

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Can we derive ISS from non-coercive ISS Lyapunov functions?

$$\dot{x} = Ax + Bu.$$

Proposition

ISS $\Leftrightarrow$ 0-UGAS $\wedge$ admissibility of B

Questions

- Lyapunov characterization of ISS
- Is it enough to consider coercive ISS Lyapunov functions?
- Characterization of integral ISS
- What is the relationship between ISS and integral ISS?

$$(\Sigma_1)\quad \begin{array}{l} \frac{\partial x}{\partial t}(z,t) = \frac{\partial^2 x}{\partial z^2}(z,t) + ax(z,t) \\ x(0,t) = 0 \quad \forall t \geq 0 \\ x(1,t) = u(t) + d(t) \quad \forall t \geq 0. \end{array} \quad \xrightarrow[\text{Volterra tr.}]{} \quad \begin{array}{l} \frac{\partial w}{\partial t}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) \\ w(0,t) = 0 \quad \forall t \geq 0 \\ w(1,t) = d(t) \quad \forall t \geq 0, \end{array} \quad (\Sigma_2)$$

Σ_1 is transformed into Σ_2 by means of

- $w(z,t) = x(z,t) + \int_0^z k(z,y)x(y,t)dy$
- $u(t) = -\int_0^1 k(1,y)x(y,t)dy$

and we naturally come to the question of ISS of Σ_2.

Questions

- ISS of the boundary control systems
- Robust stabilization of PDE systems with boundary controls
- Boundary interconnections of PDE systems

We have:

- Small-gain theorems in Lyapunov form for interconnections of n systems with in-domain inputs

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  A major difficulty here is that ISS $\neq$ AG $\wedge$ UGS

4. Stability of interconnections

We have:

- Small-gain theorems in Lyapunov form for interconnections of n systems with in-domain inputs

We would like to have:

- Small-gain theorems in trajectory form
  A major difficulty here is that ISS $\neq$ AG $\wedge$ UGS
- SGT for boundary interconnected PDEs
- SGT for infinite interconnections

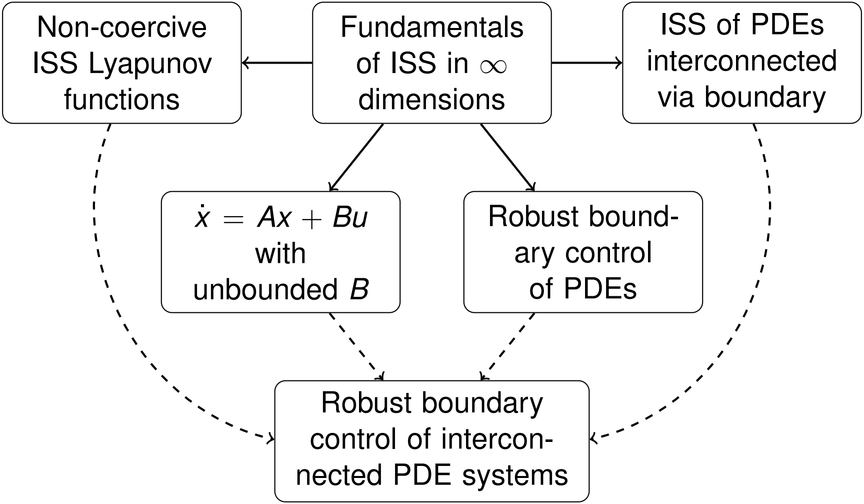
5. Construction of LFs for wide classes of systems

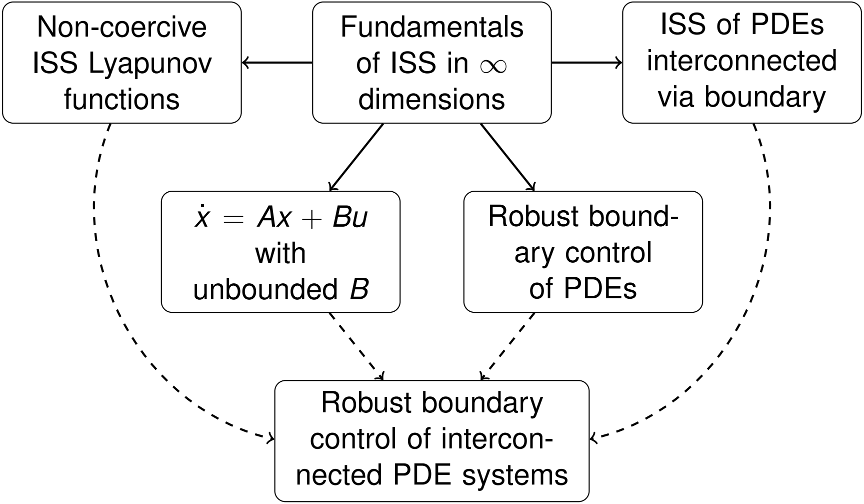
Important for applications

- ISS stabilization
- Construction of nonlinear observers
- etc.

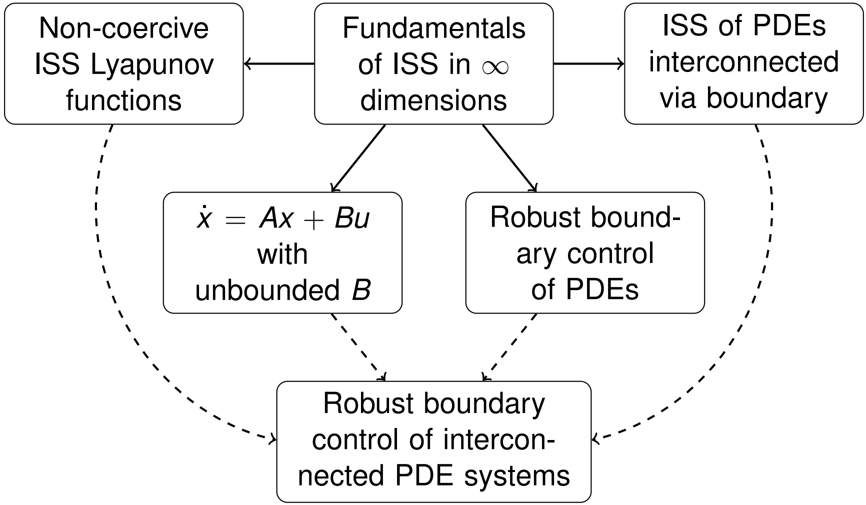
- Till now almost exclusively parabolic PDEs have been studied.
- A few results on systems of conservation laws and higher-order equations.

- No results for wave equations
- No results for nonlinear systems with boundary inputs
- No results ISS of feedbacks of different kinds of systems (ODE-PDE, heat-wave etc.)


Non-coercive ISS Lyapunov functions
Fundamentals of ISS in ∞ dimensions
ISS of PDEs interconnected via boundary
$\dot{x} = Ax + Bu$ with unbounded B
Robust boundary control of PDEs
Robust boundary control of interconnected PDE systems



Thank you for attention!



Papers and slides can be found at
www.mironchenko.com