Input-to-state stability of infinite-dimensional systems: characterizations and counterexamples

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joint work with

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Class of systems

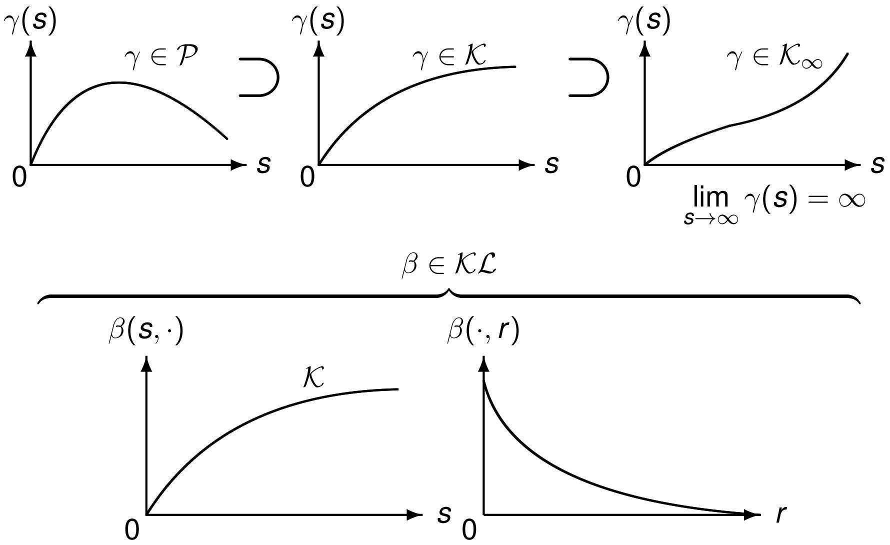
$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), & x(t) \in D(A) \subset X, \\ x(0) = x_0. \end{cases}$$

- $\mathcal{U} = PC(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.
- T is a C_0-semigroup.
- f is a Lipschitz continuous perturbation.

$x \in C([0, T], X)$ is a mild solution iff

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s), u(s))ds.$$

Comparison functions

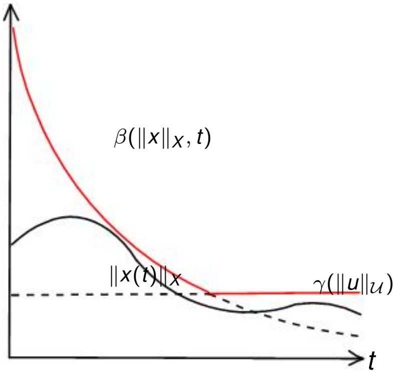


Input-to-state stability

Definition (ISS)

ISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall x \in X, \forall u \in \mathcal{U}$

$$\|\phi(t,x,u)\|_X \leq \max\Big\{\beta(\|x\|_X, t), \underbrace{\gamma}_{Gain}\Big(\sup_{s\in[0,t]} \|u(s)\|_U\Big)\Big\}.$$

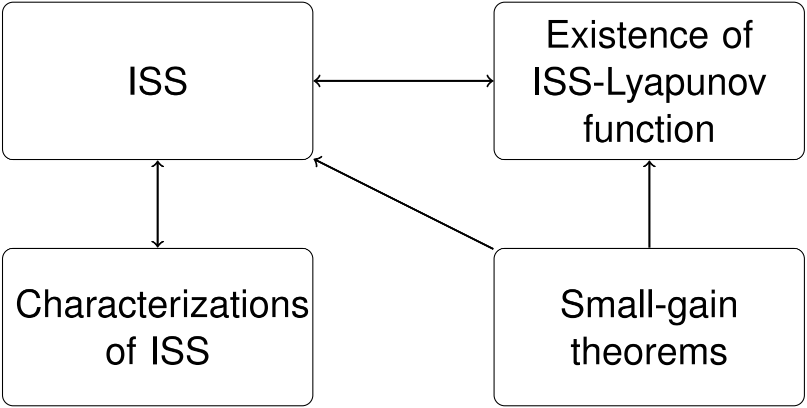


Motivation

Why ISS?

1. Unified theory of internal and external stability
2. Robust stabilization of nonlinear systems
   - M. Krstić, I. Kanellakopoulos, P. Kokotović. Nonlinear and adaptive control design, Wiley, 1995.
3. Design of robust nonlinear observers
4. Stability of networks of nonlinear control systems
   - Z.-P. Jiang, A. Teel, L. Praly. Small-gain theorem for ISS systems and applications, MCSS, 1994.
   - S. Dashkovskiy, B. Rüffer, F. Wirth. Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions, SICON, 2010.
5. ...

Fundamentals of ISS-Theory for ODEs



LISS and UGAS

Definition (GAS uniform w.r.t. state (0-UGASs))

0-UGASs $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}: \quad \forall x \in X, \forall t \geq 0$

$$\|\phi(t,x,0)\|_X \leq \beta(\|x\|_X, t).$$

Definition (LISS)

LISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty, r > 0:$

$t \geq 0, \ \|x\|_X \leq r, \ \|u\|_{\mathcal{U}} \leq r \quad \Rightarrow$

$$\|\phi(t,x,u)\|_X \leq \max\Big\{\beta(\|x\|_X, t), \underbrace{\gamma}_{Gain}\big(\sup_{s\in[0,t]} \|u(s)\|_U\big)\Big\}.$$

ISS Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),$$

Definition

$V : X \to \mathbb{R}_+$ is a non-coercive ISS-Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \le \psi_2(\|x\|_X) \quad \forall x \neq 0$

(ii) $\dot{V}_u(x) \le -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \quad \forall x \in X, \forall u \in \mathcal{U}$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}\big(V(\phi(t, x, u)) - V(x)\big).$$

V is called a coercive ISS-Lyapunov function if

$$\exists \psi_1, \psi_2 \in \mathcal{K}_\infty : \quad \psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X), \quad \forall x \neq 0.$$

Theorem (Direct Lyapunov theorem)

$\exists$ a coercive ISS Lyapunov function $\Rightarrow$ ISS.

Example of nonexponentially ISS system

$$\begin{cases} \frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial l^2} + ax - x\left(\frac{\partial x}{\partial l}\right)^2 + u, \quad \forall t > 0, \\ x(0,t) = x(\pi,t) = 0. \end{cases}$$

Different choices of input space are possible!

Proposition

Let $X = H_0^1(0,\pi)$.

1. If $U = W_0^{1,\frac{4}{3}}(0,\pi) \Rightarrow$ ISS iff $a \leq 1$.
2. If $U = L_2(0,\pi) \Rightarrow$ ISS provided $a < 1$.

$$V(x) = \int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl = \|x\|^2_{H_0^1(0,\pi)}.$$

The derivative of V along the trajectories is equal

$$\dot{V}(x) = \underbrace{-2\int_0^\pi \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl}_{Linear\ \ dynamics} + 2a\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl$$

$$\underbrace{-\frac{2}{3}\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^4 dl}_{Nonlinear\ \ dynamics} - 2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl.$$

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$$\underbrace{-\frac{2}{3}\int_0^{\pi} \left(\frac{\partial x}{\partial l}\right)^4 dl}_{Nonlinear\ \ dynamics} - 2\int_0^{\pi} \frac{\partial^2 x}{\partial l^2} u dl.$$

$$U = L_2(0,\pi) \ \Rightarrow\ -2\int_0^{\pi} \frac{\partial^2 x}{\partial l^2} u dl \le 2\omega \int_0^{\pi} \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl + 2\frac{1}{\omega}\int_0^{\pi} u^2 dl$$

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$$U = L_2(0,\pi) \quad \Rightarrow \quad -2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl \leq 2\omega \int_0^\pi \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl + 2\frac{1}{\omega}\int_0^\pi u^2 dl,$$

$$U = W_0^{1,\frac{4}{3}}(0,\pi) \quad \Rightarrow \quad -2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl \leq \frac{\omega}{4}\int_0^\pi \left|\frac{\partial x}{\partial l}\right|^4 dl + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{4}\int_0^\pi \left|\frac{\partial u}{\partial l}\right|^{\frac{4}{3}} dl$$

Case $U = L_2(0,\pi)$

$$\dot{V}(x) \leq -2(1-a-\omega)V(x) - \frac{2}{3\pi}V^2(x) + 2\frac{1}{\omega}\int_0^\pi u^2 dl.$$

- $a < 1 \Rightarrow$ ISS for $U = L_2(0,\pi)$.
- $a = 1 \Rightarrow$ 0-UGASs for $U = L_2(0,\pi)$.
- $a = 1 \Rightarrow$ Is it ISS?

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Case $U = W_0^{1,\frac{4}{3}}(0,\pi)$

$$\dot{V}(x) \leq -2(1-a)V(x) - \left(\frac{1}{3} - \frac{\omega}{2}\right)\frac{1}{\pi}V^2(x) + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{2}\|u\|^{\frac{4}{3}}_{W_0^{1,\frac{4}{3}}(0,\pi)}.$$

- $a \leq 1 \Rightarrow$ ISS for $U = W_0^{1,\frac{4}{3}}(0,\pi)$

Case $U = L_2(0,\pi)$

$$\dot{V}(x) \leq -2(1-a-\omega)V(x) - \frac{2}{3\pi}V^2(x) + 2\frac{1}{\omega}\int_0^\pi u^2 dl.$$

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- $a = 1 \Rightarrow$ 0-UGASs for $U = L_2(0,\pi)$.
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Case $U = W_0^{1,\frac{4}{3}}(0,\pi)$

$$\dot{V}(x) \leq -2(1-a)V(x) - \left(\frac{1}{3} - \frac{\omega}{2}\right)\frac{1}{\pi}V^2(x) + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{2}\|u\|^{\frac{4}{3}}_{W_0^{1,\frac{4}{3}}(0,\pi)}.$$

- $a \leq 1 \Rightarrow$ ISS for $U = W_0^{1,\frac{4}{3}}(0,\pi)$

Σ is no more 0-UGASs for $a > 1$.

In-domain couplings of parabolic systems

- Using Lyapunov small-gain theorems, we can treat ISS of nonlinear PDE systems with in-domain couplings.
- For in-domain interconnections of nonlinear parabolic ISS systems L_p setting works well.
- If some of the systems are iISS, Sobolev spaces should be used.
- Thus: choice of state space matters!

- AM, H. Ito. Construction of Lyapunov functions for interconnected parabolic systems: an iISS approach, SICON, 2015.

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),$$

Now several questions arise:

1. ISS ??? $\Rightarrow$??? $\exists$ a coercive ISS Lyapunov function
   For ODEs positively answered by Sontag, Wang in 1995 (Systems & Control Letters).

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2. What about local ISS?

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2. What about local ISS?
3. What about other characterizations of ISS?

LISS = 0-UASs = ∃ coercive LISS-LF

$$\dot{x}(t) = Ax(t) + f(x(t), u(t))$$

Theorem (AM, Sys. & Cont. Lett., 2016)

(i) $\forall C > 0\ \exists K(C) > 0$:

$$\|x\|_X \le C,\ \|y\|_X \le C \quad \Rightarrow \quad \|f(y,v) - f(x,v)\|_X \le L_f(C)\|y - x\|_X.$$

(ii) $f(x,\cdot)$ *be continuous for all* $x \in X$.

(iii) $\exists\ \sigma \in \mathcal{K}$ *and* $\rho > 0$:

$$\|v\|_U \le \rho,\ \|x\|_X \le \rho \quad \Rightarrow \quad \|f(x,v) - f(x,0)\|_X \le \sigma(\|v\|_U).$$

$$\Downarrow$$

0-UASs $\Leftrightarrow$ $\exists$ *0-UAS LF* $\Leftrightarrow$ $\exists$ *LISS LF* $\Leftrightarrow$ *LISS*

Counterexample: 0-UGAS but not LISS

$$\dot{x}_k(t) = -\frac{1}{1 + k|u(t)|} x_k(t)$$

- $X = l_1 := \{(x_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |x_k| < \infty\}$
- $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$

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- $\forall u,\ \forall x \quad \Rightarrow \quad \|\phi(t, x, u)\|_X \to 0, t \to \infty$

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But property (iii) does not hold!

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It is not LISS!

Last remarks concerning LISS

$\mathbb{R}^n$-world

1. In $\mathbb{R}^n$ (ii) $\wedge$ (i) $\Rightarrow$ (iii).

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4. Our result is more general and uses other technique.

Open question

1. If (iii) is dropped, will converse LISS Lyapunov theorem still hold, even though 0-UAS $\neq$ LISS in general?

Outline

Converse Lyapunov theorems for global ISS

Converse Lyapunov theorems for the global ISS property are more complicated!

Our plan

1. UGAS = $\exists$ UGAS Lyapunov function
2. ISS = UGAS under suitable feedback

ISS and Robust Stability

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X \qquad (\Sigma)$$

- Let $u(t) := d(t)\varphi(x(t))$.
- with $d \in \mathcal{D} = \{d : \mathbb{R}_+ \to D\}, \quad D = \{d \in U : \|d\|_U \leq 1\}$.

Then

$$\dot{x}(t) = Ax(t) + \underbrace{f(x(t), d(t)\varphi(x(t)))}_{g(x(t),d(t))}. \qquad (\tilde{\Sigma})$$

Definition

Σ is weakly uniformly robustly asymptotically stable (WURS), if $\exists$ Lipschitz continuous $\varphi : X \to \mathbb{R}_+$ and $\psi \in \mathcal{K}_\infty$: $\varphi(x) \geq \psi(\|x\|_X)$ and $\tilde{\Sigma}$ is UGAS over $\mathcal{D} = \{d : \mathbb{R}_+ \to D\}$.

Theorem

$$\dot{x}(t) = Ax(t) + g(x(t), d(t)). \qquad (\tilde{\Sigma})$$

If

- *g is Lipschitz continuous on bounded balls, uniformly w.r.t. the second argument*
- *$\tilde{\Sigma}$ is UGAS*

$\Rightarrow \exists$ *Lipschitz continuous Lyapunov function for $\tilde{\Sigma}$.*

ISS and Robust Stability

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$$\dot{x}(t) = Ax(t) + \underbrace{f(x(t), d(t)\varphi(x(t)))}_{g(x(t),d(t))}. \qquad (\tilde{\Sigma})$$

Theorem

Σ is ISS $\Rightarrow$ Σ is WURS.

$$\begin{array}{ccc} \Sigma \text{ is ISS} & \Longleftarrow & \exists \text{ ISS-LF for } \Sigma \\ \Downarrow & & \Uparrow \\ \tilde{\Sigma} \text{ is UGAS} & \Longrightarrow & \exists \text{ UGAS-LF for } \tilde{\Sigma} \end{array}$$

Outline

Definition (Stability-like notions)

$$\text{ULS} \;:\Leftrightarrow\; \exists \sigma, \gamma \in \mathcal{K}_\infty \text{ and } r > 0\text{: } \forall t \geq 0$$

$$\|x\|_X \leq r,\ \|u\|_{\mathcal{U}} \leq r \;\Rightarrow\; \|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{UGS} \;:\Leftrightarrow\; \text{ULS with } r = \infty.$$

$$\text{0-UGS} \;:\Leftrightarrow\; \exists \sigma \in \mathcal{K}_\infty : \forall t \geq 0,\ \forall x \in X$$

$$\|\phi(t,x,0)\|_X \leq \sigma(\|x\|_X)$$

Definition (Attractivity for zero inputs)

$$\text{0-GATT} \;:\Leftrightarrow\; \forall x \in X \;\Rightarrow\; \lim_{t\to\infty} \|\phi(t,x,0)\|_X = 0.$$

$$\text{0-UGATT} \;:\Leftrightarrow\; \forall \varepsilon, \delta > 0\ \exists T = T(\varepsilon,\delta) :$$

$$\forall t \geq T,\ \forall x \in X : \|x\|_X \leq \delta \;\Rightarrow\; \|\phi(t,x,0)\|_X \leq \varepsilon.$$

$$\text{0-GAS} \;:\Leftrightarrow\; \text{0-ULS + 0-GATT}.$$

Definition (Attractivity-like notions)

$$\text{AG} \;:\Leftrightarrow\quad \exists \gamma \in \mathcal{K}_\infty \cup \{0\}: \forall x \in X,\ \forall u \in \mathcal{U}$$

$$\limsup_{t\to+\infty} \|\phi(t,x,u)\|_X \le \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{AG} \;:\Leftrightarrow\quad \exists \gamma \in \mathcal{K}_\infty: \forall x \in X, \forall u \in \mathcal{U}, \forall \varepsilon > 0\ \exists T = T(\varepsilon, x, u):$$

$$t \ge T \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{sAG} \;:\Leftrightarrow\quad \exists \gamma \in \mathcal{K}_\infty: \forall x \in X, \forall \varepsilon > 0\ \exists T = T(\varepsilon, x):$$

$$t \ge T,\ u \in \mathcal{U} \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{UAG} \;:\Leftrightarrow\quad \exists \gamma \in \mathcal{K}_\infty \cup \{0\}: \forall \varepsilon, \delta > 0\ \exists T = T(\varepsilon, \delta):$$

$$t \ge T,\ u \in \mathcal{U},\ \|x\|_X \le \delta \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

Motivation

$$\dot{x} = f(x, u) \qquad \text{(ODE)}$$

Theorem (Characterizations of ISS (Sontag, Wang))

For (ODE) *the following statements are equivalent:*

1. *ISS*
2. $\exists$ *ISS-LF*
3. *UAG*
4. *AG + UGS*
5. *AG + 0-UGASs*
6. *AG + LISS*

- E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. Sys. & Cont. Letters, 1995.
- E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. IEEE TAC, 1996.

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Theorem (Characterizations of ISS (Sontag, Wang))

For (ODE) *the following statements are equivalent:*

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What does remain from this picture
in ∞ dimensions?

Counterexample 1: Linear systems

0-GAS $\not\Rightarrow$ 0-UGASs

$$\dot{x}_k = -\frac{1}{k} x_k, \quad k = 1, \ldots, \infty.$$

Denote $x := \{x_k\}_{k=1}^{\infty}$ and let

$$X = c_0 := \{x = \{x_k\}_{k=1}^{\infty} : \lim_{k\to\infty} x_k = 0 \text{ and } \|x\|_{c_0} := \sup_{k\geq 1} |x_k| < \infty\}.$$

Counterexample 2: Nonlinear systems without inputs

0-GATT $:\Leftrightarrow \quad \forall x \in X \quad \lim_{t\to\infty} \|\phi(t,x,0)\|_X = 0.$

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For linear systems $\dot{x} = Ax$

- 0-GATT $\quad \Rightarrow \quad \|\phi(t,x,0)\|_X \leq M\|x\|_X$ for some $M > 0$.

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$$X = l_2 := \{(z_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |z_i|^2 < \infty\}, \quad z_i = (x_i, y_i) \in \mathbb{R}^2.$$

$$\Sigma : \begin{cases} \Sigma_i : \begin{cases} \dot{x}_i = -x_i + x_i^2 y_i - \frac{1}{i^2} x_i^3, \\ \dot{y}_i = -y_i. \end{cases} \\ i = 1, \ldots, \infty \end{cases}$$

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- 0-GATT $\quad \Rightarrow \quad \|\phi(t,x,0)\|_X \leq M\|x\|_X$ for some $M > 0$.

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- Σ is uniformly (locally) exponentially stable
- Σ is 0-GAS
- Σ is not 0-UGS

For undisturbed systems we have the complete picture!

Undisturbed infinite-dimensional systems

For undisturbed systems we have the complete picture!

$$\text{0-UGAS} \Longleftrightarrow \exists \text{ of 0-UGAS-LF}$$

$$(1) \Downarrow \qquad\qquad (1)(2) \Downarrow$$

$$\text{0-GAS} \wedge \text{0-UGS} \quad \begin{matrix} \not\Rightarrow \\ \not\Leftarrow \end{matrix} \quad \text{0-GAS} \wedge \text{0-UAS}$$

$$(1)(2) \searrow \qquad\qquad \swarrow (1)$$

$$\text{0-GAS}$$

Implications become equivalences for

1. ODE systems
2. Linear systems

Counterexample 3: Nonlinear systems with inputs

$$\dot{x}_k(t) = -\frac{1}{1 + |u(t)|^k} x_k(t)$$

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But it is not ISS!

And it is not sAG with zero gain!

ISS = UAG

Σ is robustly forward complete (RFC):

$$R > 0,\ \tau > 0 \quad \Rightarrow \quad \sup_{\|x\|_X \leq R,\ t \in [0,\tau]} \|\phi(t,x)\|_X < \infty.$$

$0 \in X$ is a robust equilibrium point of Σ, if

- $\phi(t,0,d) = 0 \quad \forall t \geq 0,\ \forall d \in \mathcal{D}$.
- $\forall \varepsilon > 0, \forall h > 0 \ \ \exists \delta = \delta(\varepsilon,h) > 0$:

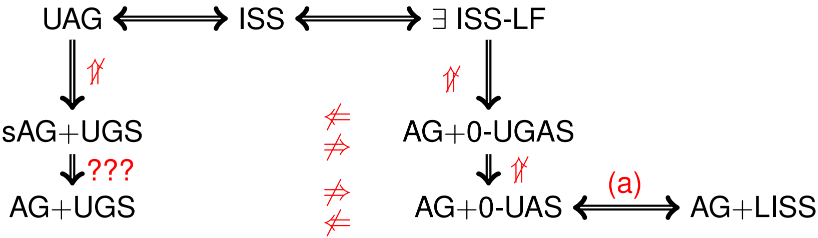
$$t \in [0,h],\ \|x\|_X \leq \delta \quad \Rightarrow \quad \|\phi(t,x)\|_X \leq \varepsilon.$$

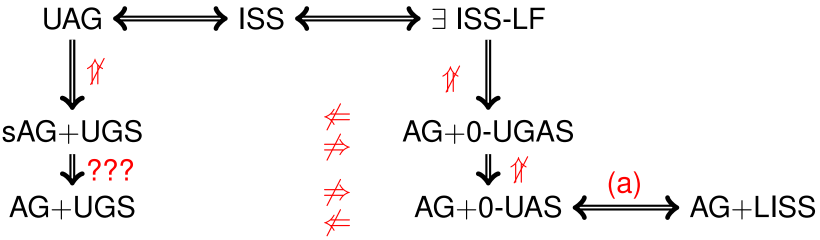
Theorem

- *Σ be RFC*
- *0 be a robust equilibrium of Σ*

$\Rightarrow$ *ISS* $\Leftrightarrow$ *UAG*

Characterizations of ISS

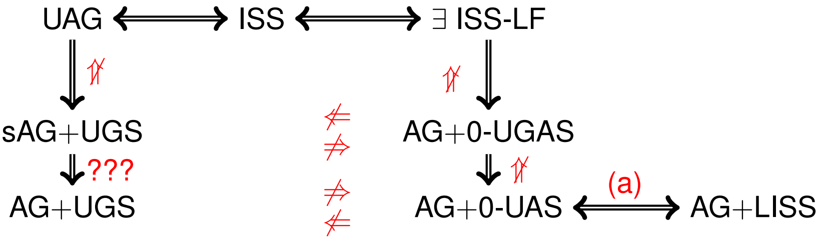




Open questions

- Relations between AG, sAG, UAG are still unknown
  - Is sAG $\Leftrightarrow$ AG ?
  - Is sISS $\Leftrightarrow$ AG $\wedge$ UGS ?

Characterizations of ISS



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  - strong ISS = ISS for finite-dimensional systems
  - strong ISS = strong stability of T for linear undisturbed infinite-dimensional systems.

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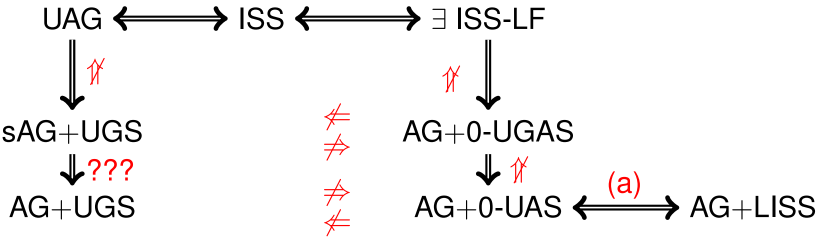

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- It makes sense to introduce "strong ISS" property:
  - strong ISS = ISS for finite-dimensional systems
  - strong ISS = strong stability of T for linear undisturbed infinite-dimensional systems.
- How to characterize strong ISS?

Characterization of strong ISS

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X \qquad (\Sigma)$$

Definition

Σ is strongly input-to-state stable (sISS), if $\exists \gamma \in \mathcal{K}$, $\sigma \in \mathcal{K}_\infty$ and $\beta : X \times \mathbb{R}_+ \to \mathbb{R}_+$:

1. $\beta(x, \cdot) \in \mathcal{L}$ for all $x \in X$, $x \neq 0$
2. $\beta(x, t) \leq \sigma(\|x\|_X)$ for all $x \in X$ and all $t \geq 0$
3. for all $x \in X$, all $u \in \mathcal{U}$ and all $t \geq 0$ it holds that

$$\|\phi(t, x, u)\|_X \leq \beta(x, t) + \gamma(\|u\|_{\mathcal{U}}).$$

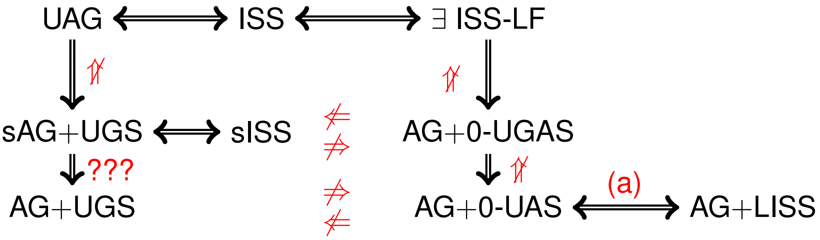
Theorem

The following statements are equivalent.

(i) *Σ is sISS.*

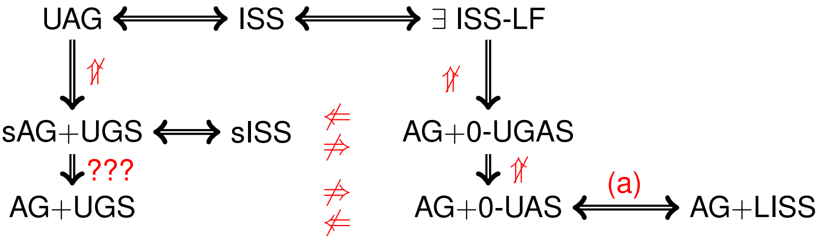
(ii) *Σ is sAG and UGS.*

Summary



- A.M. *Local input-to-state stability: Characterizations and counterexamples.* Sys. & Con. Lett., 2016
- A.M., F. Wirth. *Restatements of input-to-state stability in infinite dimensions: what goes wrong?* Proc. of the MTNS 2016
- A.M., F. Wirth. *Global converse Lyapunov theorems for infinite-dimensional systems.* Accepted to NOLCOS 2016

Summary



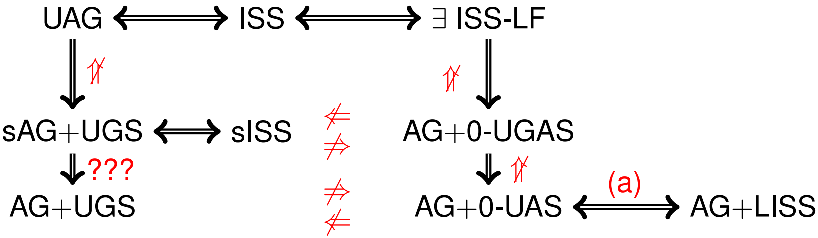
What we already know

- Converse Lyapunov theorem for LISS
- Converse Lyapunov theorem for ISS
- Nonuniform notions $\not\Rightarrow$ Uniform notions
- There are several nonequivalent groups of notions.

What we do not know:

- Difference between AG, sAG and UAG

Summary



What we do not know:

- Difference between AG, sAG and UAG
- Characterizations of strong ISS.

Outline

1. Non-coercive Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t)),$$

$V : X \to \mathbb{R}_+$ is a non-coercive ISS-Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \neq 0$

(ii) $\dot{V}(x) \leq -\alpha(\|x\|_X) \quad \forall x \in X$,

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Theorem (AM, F. Wirth, NOLCOS 2016)

Let:

- Σ *be RFC*
- 0 *be a robust equilibrium of* Σ

If V is a noncoercive Lyapunov function for Σ $\Rightarrow$ Σ is UGAS.

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Theorem (AM, F. Wirth, NOLCOS 2016)

Let:

- *Σ be RFC*
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If V is a noncoercive Lyapunov function for $\Sigma \quad \Rightarrow \quad \Sigma$ is UGAS.

Can we derive ISS from non-coercive ISS Lyapunov functions?

$$\dot{x} = Ax + Bu.$$

Proposition

ISS $\Leftrightarrow$ 0-UGAS + admissibility of B

Questions

- Lyapunov characterization of ISS
- Is it enough to consider coercive Lyapunov functions?
- Characterization of integral ISS
- What is the relationship between ISS and integral ISS?

3. Robust boundary control of PDEs

$$(\Sigma_1)\quad \begin{array}{l} \frac{\partial x}{\partial t}(z,t) = \frac{\partial^2 x}{\partial z^2}(z,t) + ax(z,t) \\ x(0,t) = 0 \quad \forall t \geq 0 \\ x(1,t) = u(t) + d(t) \quad \forall t \geq 0. \end{array} \quad \xrightarrow[\text{Volterra tr.}]{} \quad \begin{array}{l} \frac{\partial w}{\partial t}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) \\ w(0,t) = 0 \quad \forall t \geq 0 \\ w(1,t) = d(t) \quad \forall t \geq 0, \end{array} \quad (\Sigma_2)$$

Σ_1 is transformed into Σ_2 by means of

- $w(z,t) = x(z,t) + \int_0^z k(z,y)x(y,t)dy$
- $u(t) = -\int_0^1 k(1,y)x(y,t)dy$

and we naturally come to the question of ISS of Σ_2.

Questions

- ISS of the boundary control systems
- Robust stabilization of PDE systems with boundary controls
- Boundary interconnections of PDE systems

We have:

- Small-gain theorems in Lyapunov form for interconnections of n systems with in-domain inputs

4. Stability of interconnections

We have:

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We would like to have:

- Small-gain theorems in trajectory form

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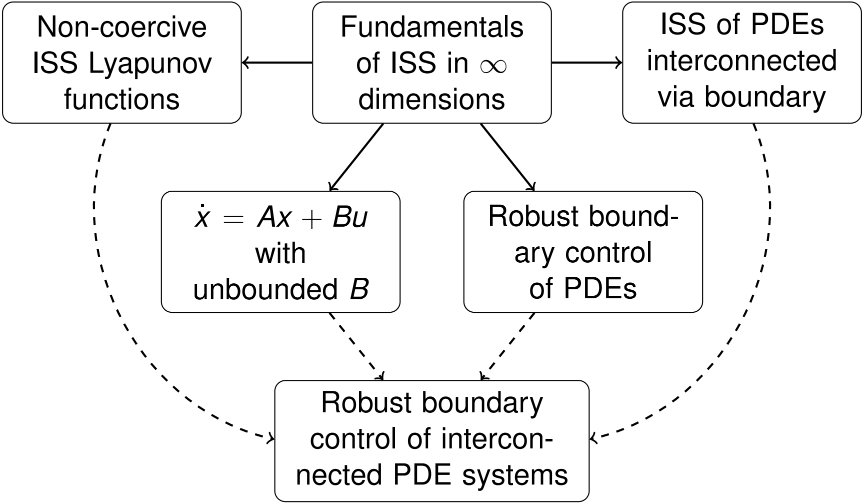
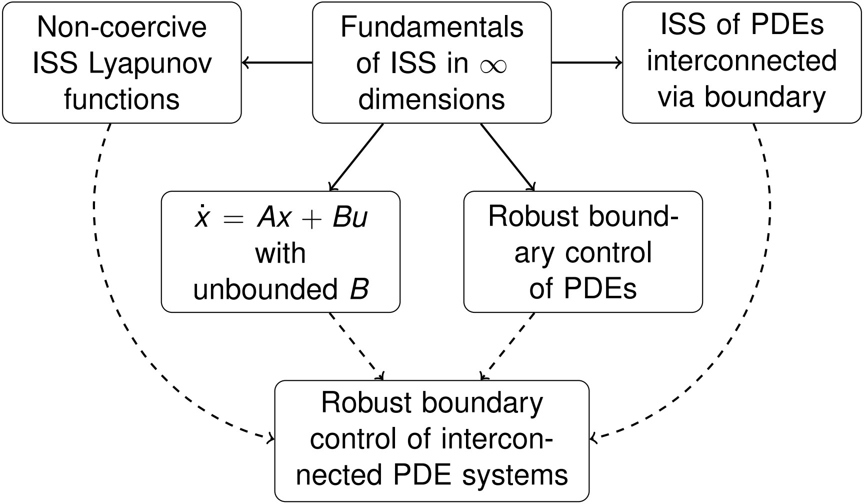
- SGT for boundary interconnected PDEs


Non-coercive ISS Lyapunov functions
Fundamentals of ISS in ∞ dimensions
ISS of PDEs interconnected via boundary
$\dot{x} = Ax + Bu$ with unbounded B
Robust boundary control of PDEs
Robust boundary control of interconnected PDE systems



Thank you for attention!