Criteria for practical input-to-state stability

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joint work with:

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Class of systems

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), & x(t) \in D(A) \subset X, \\ x(0) = x_0. \end{cases}$$

- $\mathcal{U} = PC(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.
- T is a C_0-semigroup.
- f is a Lipschitz continuous perturbation.

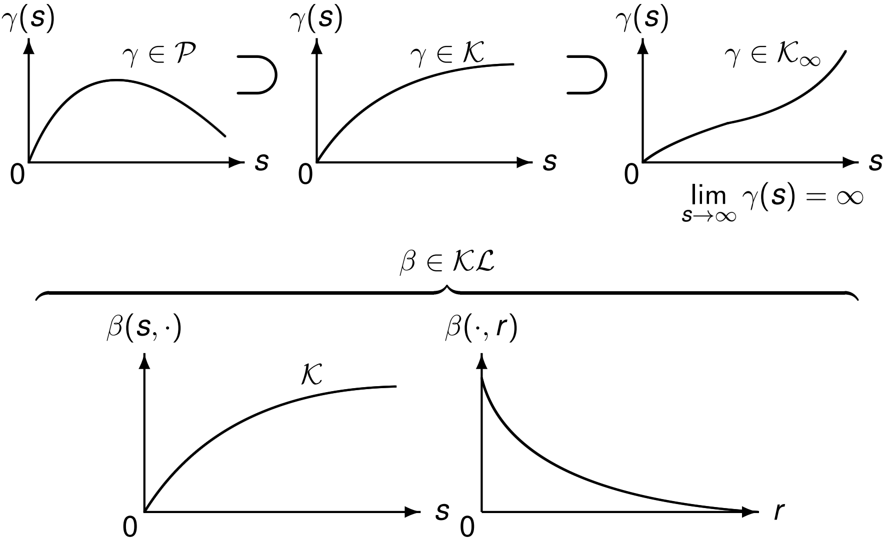
$x \in C([0, T], X)$ is a mild solution iff

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s), u(s))ds.$$

Most of results hold for much more general systems, inlcuding:

- some classes of boundary control systems
- time-delay systems
- switched systems (with ∞ number of switching modes)
- ...

Comparison functions

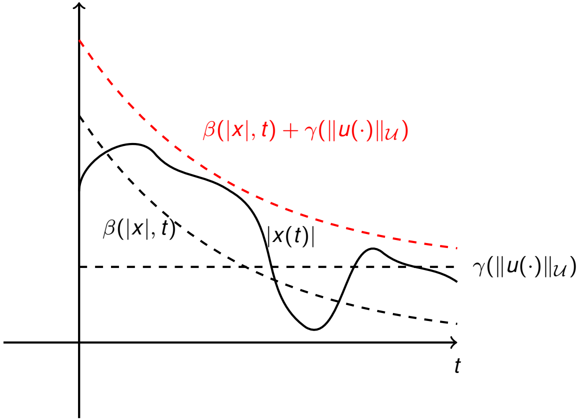


Input-to-state stability

Definition (ISS)

ISS $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$: $\forall t \geq 0, \forall x \in X, \forall u \in \mathcal{U}$

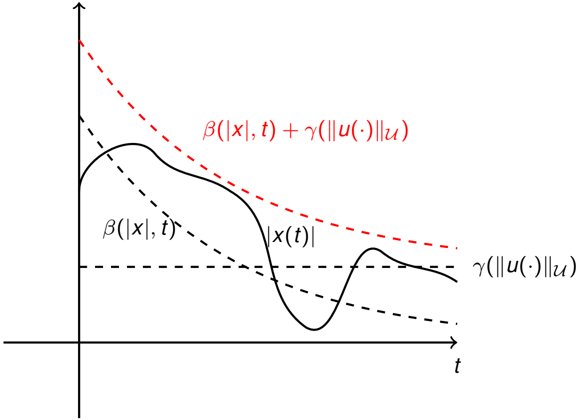
$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}).$$



Definition (Input-to-state practical stability)

ISpS $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty, \exists c > 0$: $\forall t \geq 0, \forall x \in X, \forall u \in \mathcal{U}$

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}) + c.$$



Infinite-dimensional ISS theory: 2007 – now

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$> 2/3$ of papers appeared since 2016.

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$> 2/3$ of papers appeared since 2016.

Time-delay results are not in the above list
(were already a mature subject at 2007)

Motivation

Some milestones of ISpS theory of ODE systems

1989 Sontag introduces ISS

1994 Jiang, Teel, Praly introduce ISpS and prove nonlinear small-gain theorem for couplings of 2 systems

1995,1996 Sontag, Wang show foundational characterizations of ISS as well as partially characterized ISpS.

2007 Dashkovskiy, Rüffer, Wirth prove (using characterizations of ISS) nonlinear small-gain theorem for couplings of n systems

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Recall that for ODEs

$$|\phi(t,x,0)| \leq \beta(t,|x|) \quad \Leftrightarrow \quad \text{Global asymptotic stability}$$
$$\Leftrightarrow \quad \text{Local stability} \wedge \text{Global attractivity}$$

Sontag and Wang generalized this result to the ISS setting.

LIM and ULIM

Definition

$$\text{LIM} \quad :\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty: \forall x \in X, \forall u \in \mathcal{U}, \forall \varepsilon > 0 \ \exists T = T(\varepsilon, x, u):$$

$$\|\phi(t, x, u)\|_X \leq \varepsilon + \gamma(\|u\|_\mathcal{U}).$$

$$\text{ULIM} \quad :\Leftrightarrow \quad \exists\, \gamma \in \mathcal{K}: \forall \varepsilon > 0, \forall r > 0 \quad \exists \tau = \tau(\varepsilon, r):$$

$$\|x\|_X \leq r,\ u \in \mathcal{U} \quad \Rightarrow \quad \exists t \leq \tau(r): \ \|\phi(t, x, u)\|_X \leq \varepsilon + \gamma(\|u\|_\mathcal{U}).$$

Definition

LIM $:\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty \colon \forall x \in X, \forall u \in \mathcal{U}, \forall \varepsilon > 0 \ \exists T = T(\varepsilon, x, u) :$

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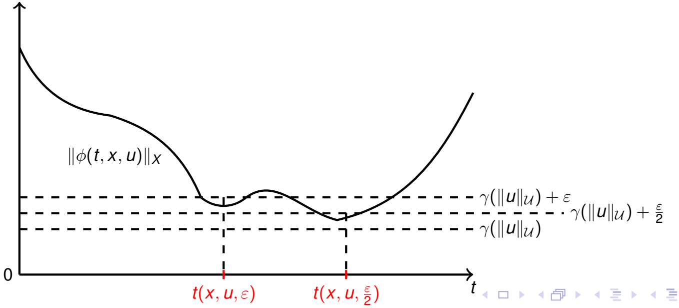

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- Limit property is a reminiscent of a so-called 'weak attractivity'.
- Limit property is closely related to the concept of recurrent sets.

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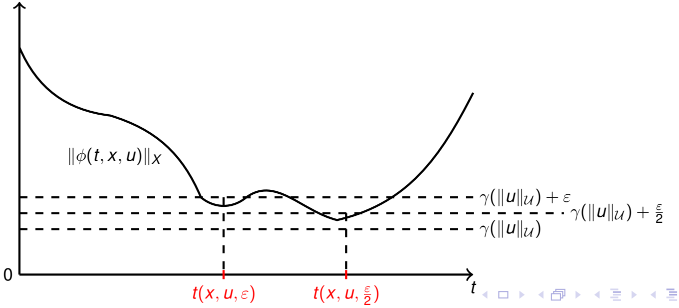

Result of Sontag and Wang

Definition (Stability)

$$\text{ULS} \;:\Leftrightarrow\; \exists r>0,\ \exists \sigma,\gamma \in \mathcal{K}_\infty:$$

$$t \geq 0, x \in B_r, u \in B_{r,\mathcal{U}} \;\Rightarrow\; \|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{0-ULS} \;:\Leftrightarrow\; \exists r>0,\ \exists \sigma \in \mathcal{K}_\infty:$$

$$t \geq 0, x \in B_r \;\Rightarrow\; \|\phi(t,x,0)\|_X \leq \sigma(\|x\|_X).$$

Theorem (Sontag, Wang, TAC 1996)

Let FC:= "forward-complete".

$$(ODE): \qquad \dot{x} = f(x,u), \quad x(t) \in \mathbb{R}^n.$$

$$(ODE) \text{ is ISS} \;\Leftrightarrow\; (ODE) \text{ is FC} \;\wedge\; (ODE) \text{ is LIM} \;\wedge\; (ODE) \text{ is 0-ULS}$$

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Proposition (Mironchenko, Wirth, TAC 2018)

(ODE) *is LIM* $\Leftrightarrow$ (ODE) *is ULIM*

Characterizations of ISS for ∞-dim systems

Characterizations due to Sontag and Wang cannot be straightforwardly transferred to ∞-dim. New notions and more uniformity is needed

Definition

Σ has bounded reachability sets (BRS), if:

$$C > 0,\ \tau > 0 \quad \Rightarrow \quad \sup_{\|x\|_X \le C,\ \|u\|_{\mathcal{U}} \le C,\ t \in [0,\tau]} \|\phi(t,x,u)\|_X < \infty.$$

Theorem (Mironchenko, Wirth, TAC 2018)

Consider a forward-complete system

$$(EE): \qquad \dot{x} = Ax + f(x,u), \quad (A, D(A)) : X \to X.$$

(EE) is ISS $\Leftrightarrow$ *(EE) is BRS* $\wedge$ *(EE) is ULIM* $\wedge$ *(EE) is 0-ULS*

Characterizations of ISS for ∞-dim systems

Importance

- Relations of ISS to other stability notions, e.g. nonlinear $L_2 \to L_2$-gain
- Basis for the proof of general small-gain theorems
- Essential for the theory of non-coercive ISS Lyapunov function
- (Hopefully) will lead to improvements of Lyapunov-Krasovskii methodology for time-delay systems
- Extensions to practical ISS
- ...

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Characterizations of ISS for ∞-dim systems

- A. M.. Local input-to-state stability: Characterizations and counterexamples. Systems & Control Letters, 87:23–28, 2016.
- A. M., F. Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. IEEE TAC, 63 (6): 1602–1617, 2018.

Application I: Non-coercive ISS LFs

Definition

$V : X \to \mathbb{R}_+$ is a non-coercive ISS Lyapunov function for $\Sigma = (X, \mathcal{U}, \phi)$, if $\exists\ \psi_2, \alpha \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$:

$$0 < V(x) \le \psi_2(\|x\|_X), \quad \forall x \in X \tag{1}$$

and

$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + \sigma(\|u\|_{\mathcal{U}}), \quad \forall x \in X, \quad u \in \mathcal{U}. \tag{2}$$

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Next we show an essentially nonlinear result.

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Theorem (Mironchenko, Wirth, IEEE 2018, Jacob, Mironchenko, Partington, Wirth, CDC 2018)

Let Σ be a forward complete control system, which is BRS and is continuous near equilibrium.
$\exists$ a noncoercive ISS Lyapunov function for Σ, then Σ is ISS.

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Proof.

- Σ is FC $\wedge$ $\exists$ nc-LF $\quad\Rightarrow\quad$ Σ is ULIM.
- Σ is FC $\wedge$ continuity $\wedge$ $\exists$ nc-LF $\quad\Rightarrow\quad$ Σ is ULS.
- Σ is BRS $\wedge$ continuity $\wedge$ $\exists$ nc-LF $\quad\Rightarrow\quad$ Σ is ISS.

$\square$

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□

Note, that we cannot resort in the proof to the comparison principle or some linear methods.

Example (Jacob, Mironchenko, Partington, Wirth, CDC 2018)

$$x_t(\xi,t) = x_{\xi\xi}(\xi,t), \quad \xi \in (0,1),\ t>0,$$
$$x(0,t) = 0, \quad x(1,t) = u(t), \quad t > 0.$$

We choose $X = L^2(0,1), \quad U = \mathbb{C}, \quad B = \delta_1'$,

$$Af = f'', \quad f \in D(A) := \Big\{ f \in H^2(0,1) \mid f(0) = f(1) = 0 \Big\}.$$

- A is a self-adjoint operator on X
- A generates an exponentially stable analytic C_0-semigroup on X
- $B \in L(U, X_{-1})$ is ∞-admissible
- A non-coercive ISS Lyapunov function is given by:

$$V(x) = -\langle A^{-1}x, x\rangle_X = \int_0^1 \left(\int_\xi^1 (\xi - \tau) x(\tau) d\tau \right) \overline{x(\xi)} d\xi$$

- No coercive ISS LFs are known for this system.

Input-to-state practical stability (ISpS)

Definition

Let $\mathcal{A} \subset X$.

ISpS wrt $\mathcal{A}$:⇔ $\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty, c > 0$:
$x \in X, t \geq 0, u \in \mathcal{U} \;\Rightarrow\; \|\phi(t,x,u)\|_{\mathcal{A}} \leq \beta(\|x\|_{\mathcal{A}}, t) + \gamma(\|u\|_{\mathcal{U}}) + c.$

ISpS :⇔ $\exists$ a bounded set $\mathcal{A} \subset X$: Σ is ISpS w.r.t. $\mathcal{A}$.

ISS wrt $\mathcal{A}$:⇔ ISpS w.r.t. $\mathcal{A}$ with $c := 0$.

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Why ISpS?

- It is often impossible or too costly to construct a feedback, which makes the system ISS
- Quantized control
- Funnel control
- ...

Application II: Criteria of ISpS

ISpS plan

- Our next aim is to obtain the criteria of ISpS
- Some of these characterizations are new (and stronger than existing ones) even for ODE systems.
- Simple application of ISS characterizations is not possible. New ideas will be needed.

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Definition

Let $\Sigma = (X, \mathcal{U}, \phi)$ be given.

- $\mathcal{A} \subset X$ is called *s-invariant* if:

$$t \geq 0,\, x \in \mathcal{A},\, \|u\|_{\mathcal{U}} \leq s \quad \Rightarrow \quad \phi(t,x,u) \in \mathcal{A}.$$

- An s-invariant set $\mathcal{A} \subset X$ is called *robustly s-invariant* if: $\forall \varepsilon > 0$, $\forall h > 0$ $\exists \delta = \delta(\varepsilon, h) > 0$:

$$t \in [0,h],\ \|x\|_{\mathcal{A}} \leq \delta,\ \|u\|_{\mathcal{U}} \leq \delta \ \Rightarrow \ \|\phi(t,x,u)\|_{\mathcal{A}} \leq \varepsilon.$$

Uniform limit property (A.M., F. Wirth, 2018)

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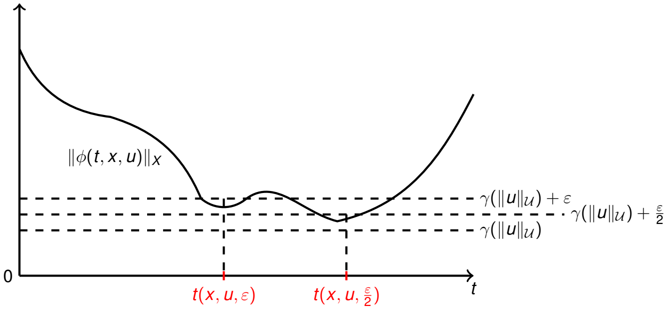

Related concepts are: (uniform) weak attractivity, recursivity.

Main ISpS Result

Theorem (AM, Accepted to TAC 2018))

Consider a BRS system (EE) satisfying

- $f : X \times U \to X$ *is Lipschitz continuous on bounded subsets of* X*, uniformly with respect to the second argument.*
- $f(x, \cdot)$ *is continuous for all* $x \in X$*.*

The following statements are equivalent:

1. Σ *is ISpS*
2. $\forall s > 0$ *there is a bounded s-invariant set* $\mathcal{A} \subset X$*:* Σ *is ISS wrt* $\mathcal{A}$*.*
3. *There is a bounded set* $\mathcal{A} \subset X$*:* Σ *is ULIM w.r.t.* $\mathcal{A}$*.*

Proving the main result: understanding ULIM

Definition

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$\mathcal{A} \subset X$ is ULIM $\quad \Rightarrow \quad$ any $\tilde{A} \supset \mathcal{A}$ is ULIM.

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Can we find $\tilde{A} \supset \mathcal{A}$ with stronger properties than ULIM?

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Proposition (AM, Accepted to IEEE TAC 2018)

Let: $\mathcal{A} \subset X$ be a bounded (not necessarily 0-invariant) set.
$\forall \varepsilon > 0$ denote
$$\mathcal{A}_{\varepsilon,\gamma} := \{\phi(t, x, u) : t \in \mathbb{R}_+,\ x \in B_\varepsilon(\mathcal{A}),\ \|u\|_{\mathcal{U}} \le \gamma^{-1}(\tfrac{\varepsilon}{2})\}.$$

Σ is BRS $\wedge$ Σ is ULIM w.r.t. $\mathcal{A}$
$\Rightarrow \quad \forall\, \varepsilon > 0 \quad \mathcal{A}_\varepsilon$ is bounded, $\gamma^{-1}(\frac{\varepsilon}{2})$-invariant and Σ is ISS w.r.t. $\mathcal{A}_\varepsilon$.

Theorem (Characterization of ISpS (AM, Accepted to TAC, 2018))

Consider a BRS system (EE) with a Lipschitz f. The following statements are equivalent:

1. Σ *is ISpS*
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This theorem can be generalized to a much more broad class of systems including:

- evolution equations in Banach spaces
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Theorem (Characterization of ISpS (AM, Accepted to TAC, 2018))

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Specialization of the results to the ODE case is also of interest.

ISpS of ODEs

$$\dot{x} = f(x, u) \qquad \text{(ODE)}$$

- $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz wrt x, uniformly wrt u
- $\mathcal{U} := L_\infty(\mathbb{R}_+, \mathbb{R}^m)$

Proposition (AM, F. Wirth, IEEE TAC 2018)

Let $\mathcal{A} \subset \mathbb{R}^n$ be a bounded set.

$$\text{(ODE) is ULIM w.r.t. } \mathcal{A} \quad \Leftrightarrow \quad \text{(ODE) is LIM w.r.t. } \mathcal{A}.$$

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Proposition (Lin, Sontag, Wang, SICON 1996)

$$\text{(ODE) is FC} \quad \Leftrightarrow \quad \text{(ODE) is BRS.}$$

ISpS of ODEs

Proposition (E. Sontag and Y. Wang, 1996)

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Theorem (AM, IEEE TAC 2018)

Let (ODE) be forward-complete. The following statements are equivalent:

1. *(ODE) is ISpS*
2. *For any $s > 0$ there is a compact s-invariant set $\mathcal{A} \subset \mathbb{R}^n$: (ODE) is ISS w.r.t. $\mathcal{A}$.*
3. *There is a bounded set $\mathcal{A} \subset \mathbb{R}^n$: (ODE) is LIM w.r.t. $\mathcal{A}$.*

How to check ULIM property?

Definition

$V : X \to \mathbb{R}_+$ is called a noncoercive ISS Lyapunov function, if there exist $\psi_2, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that

$$0 < V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in X$$

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Corollary

Σ is BRS $\wedge$ V is a non-coercive ISS-LF $\Rightarrow$ Σ is ISpS.

Overview

Outcomes

- ISS $\Leftrightarrow$ ULIM $\wedge$ BRS $\wedge$ 0-ULS
- BRS $\wedge$ continuity near equilibrium $\wedge$ non-coercive ISS LF $\Rightarrow$ ISS
- ISpS results:
  - ISpS $\Leftrightarrow$ BRS $\wedge$ $\exists$ bounded $\mathcal{A} \subset X$: Σ is ULIM w.r.t. $\mathcal{A}$
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- A. Mironchenko. Criteria for input-to-state practical stability, to appear in IEEE TAC, https://arxiv.org/abs/1710.10452
- A. Mironchenko, F. Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. IEEE TAC, 63 (6): 1602-1617, 2018.
- B. Jacob, A. Mironchenko, J. Partington and F. Wirth. Remarks on input-to-state stability and noncoercive Lyapunov functions. Accepted to CDC 2018.

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Future research

- Understanding non-coercive ISS Lyapunov functions
- Proof of general small-gain theorems in trajectory form
- Time-delay systems:
  - Relaxed LK functionals for TDS (posed by Antoine)
  - Improve ISS characterizations for TDS
  - Small-gain theorems for time-delay systems