Constructions of Lyapunov functions for nonlinear parabolic control systems: an integral ISS approach

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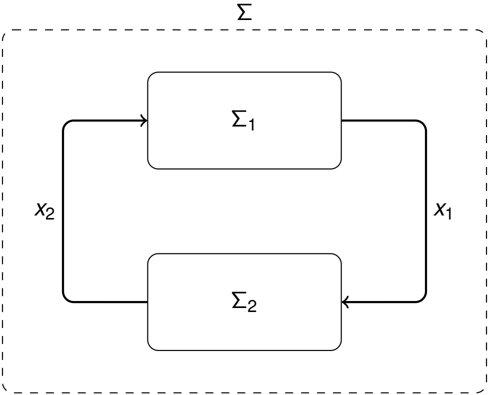
Systems under consideration



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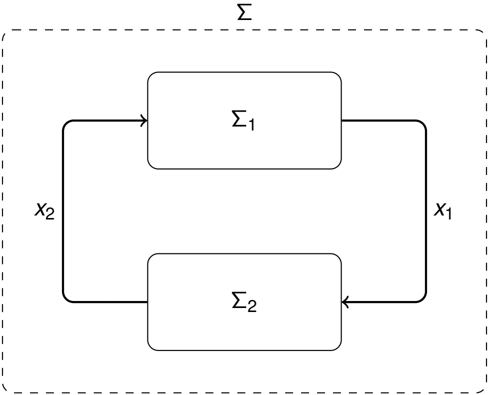


- If Σ_i are ODEs $\Rightarrow$ Theory is well-developed.
- Our interest: Σ_i - nonlinear parabolic PDEs.

Outline

1. Basic definitions
2. ISS and iISS of single nonlinear systems
3. Interconnections of iISS systems
4. Summary and Outlook

Class of systems

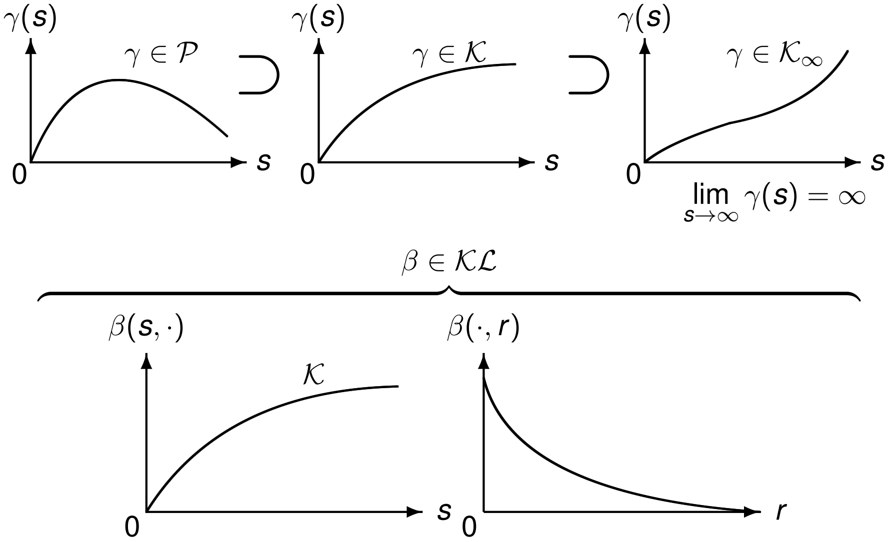
$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X, \\ x(0) = \phi_0. \end{cases}$$

- $u \in C(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$
- f is Lipschitz w.r.t. x

$x \in C([0, T], X)$ is a mild solution iff

$$x(t) = T(t)\phi_0 + \int_0^t T(t-s)f(x(s), u(s))ds.$$

Comparison functions



Input-to-state stability

Definition (GAS uniform w.r.t. state (0-UGASs))

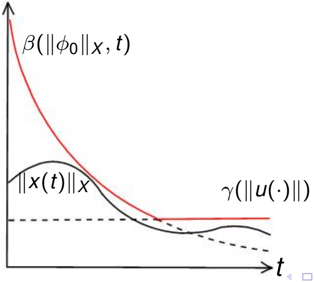
0-UGASs :⇔ $\exists \beta \in \mathcal{KL}$: $\forall \phi_0 \in X, \forall t \geq 0$

$$\|\phi(t, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t).$$

Definition (ISS)

ISS :⇔ $\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$: $\forall t \geq 0, \forall \phi_0 \in X, \forall u \in C(\mathbb{R}_+, U)$

$$\|\phi(t, \phi_0, u)\|_X \leq \max\Big\{\beta(\|\phi_0\|_X, t), \underbrace{\gamma}_{Gain}\Big(\sup_{s\in[0,t]} \|u(s)\|_U\Big)\Big\}.$$



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Definition (integral input-to-state stability (iISS))

iISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \alpha, \mu \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall \phi_0 \in X, \forall u \in C(\mathbb{R}_+, U)$

$$\alpha(\|\phi(t, \phi_0, u)\|_X) \leq \beta(\|\phi_0\|_X, t) + \int_0^t \mu(\|u(s)\|_U) ds.$$

Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),$$
$$x(0) = \phi_0,$$

Definition

$V : X \to \mathbb{R}_+$ is an iISS-Lyapunov function iff $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty,\ \sigma, \alpha \in \mathcal{K}$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $\dot{V}_u(x) \leq -\alpha(V(x)) + \sigma(\|u(0)\|_U)$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x)).$$

If $\alpha \in \mathcal{K}_\infty$, then V is called an ISS-Lyapunov function.

Theorem

$\exists$ *ISS/iISS Lyapunov function* $\Rightarrow$ *ISS/iISS.*

Outline-II

We want to interconnect ISS and iISS systems.

Decompose Σ into Σ_1, Σ_2

↓

Choose state spaces for Σ_1, Σ_2

↓

Find ISS/iISS Lyapunov functions for Σ_1, Σ_2

↓

Apply iISS small-gain theorem

↓

iISS

Outline-II

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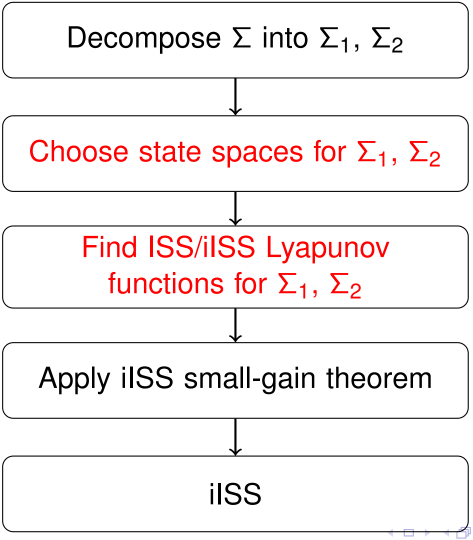

Candidate for iISS: bilinear systems

$$\dot{x}(t) = Ax(t) + Bu(t) + C(x(t), u(t)),$$

where $B \in L(U, X)$, and $\exists K > 0$:

$$\|C(x,u)\|_X \leq K \|x\|_X \|u\|_U.$$

The simplest example:

$$\dot{x} = -x + xu.$$

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For $V(x) = \ln(1 + x^2)$ we have

$$\dot{V}(x) = \frac{2x\dot{x}}{1+x^2} = -\frac{2x^2}{1+x^2} + \frac{2x^2 u}{1+x^2} \leq -\frac{2x^2}{1+x^2} + 2|u| = -(1-e^{-V(x)}) + 2|u|.$$

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Theorem (E. Sontag, 1998)

Finite-dimensional bilinear 0-GAS systems are iISS.

Criteria for iISS of bilinear systems

$$\dot{x}(t) = Ax(t) + Bu(t) + C(x(t), u(t)). \qquad \text{(BL)}$$

Theorem (A. Mironchenko, H. Ito)

X - Banach; B, C - bounded $\Rightarrow$ *TFAE:*

- (BL) *is iISS*
- (BL) *is 0-UGASs*
- $W : x \mapsto \ln\left(1 + \int_0^\infty \|T(t)x\|^2 dt\right)$ *is a (in general) non-coercive iISS Lyapunov function for* (BL).

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X - Hilbert $\Rightarrow$ $W(x) = \ln\left(1 + \langle Px, x\rangle\right)$,

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle = -\|x\|_X^2, \quad \forall x \in D(A).$$

Example of nonexponentially ISS system

$$\begin{cases} \frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial l^2} + ax - x\left(\frac{\partial x}{\partial l}\right)^2 + u, & \forall t > 0, \\ x(0,t) = x(\pi,t) = 0. \end{cases}$$

Different choices of input space are possible!

Proposition

Let $X = H_0^1(0,\pi)$.

1. If $U = W_0^{1,\frac{4}{3}}(0,\pi) \Rightarrow$ ISS iff $a \leq 1$.
2. If $U = L_2(0,\pi) \Rightarrow$ ISS provided $a < 1$.

$$V(x) = \int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl = \|x\|^2_{H^1_0(0,\pi)}.$$

The derivative of V along the trajectories is equal

$$\dot{V}(x) = \underbrace{-2\int_0^\pi \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl}_{Linear\ \ dynamics} + 2a\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl$$

$$\underbrace{-\frac{2}{3}\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^4 dl}_{Nonlinear\ \ dynamics} - 2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl.$$

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$$U = L_2(0,\pi) \;\; \Rightarrow \;\; -2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl \leq 2\omega \int_0^\pi \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl + 2\frac{1}{\omega}\int_0^\pi u^2 dl$$

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$$U = W_0^{1,\frac{4}{3}}(0,\pi) \Rightarrow -2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl \leq \frac{\omega}{4}\int_0^\pi \left|\frac{\partial x}{\partial l}\right|^4 dl + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{4}\int_0^\pi \left|\frac{\partial u}{\partial l}\right|^{\frac{4}{3}} dl$$

Case $U = L_2(0,\pi)$

$$\dot{V}(x) \leq -2(1-a-\omega)V(x) - \frac{2}{3\pi}V^2(x) + 2\frac{1}{\omega}\int_0^{\pi} u^2 dl.$$

- $a < 1 \Rightarrow$ ISS for $U = L_2(0,\pi)$.
- $a = 1 \Rightarrow$ 0-UGASs for $U = L_2(0,\pi)$.
- $a = 1 \Rightarrow$ Is it ISS?

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Case $U = W_0^{1,\frac{4}{3}}(0,\pi)$

$$\dot{V}(x) \leq -2(1-a)V(x) - \left(\frac{1}{3} - \frac{\omega}{2}\right)\frac{1}{\pi}V^2(x) + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{2}\|u\|^{\frac{4}{3}}_{W_0^{1,\frac{4}{3}}(0,\pi)}.$$

- $a \leq 1 \Rightarrow$ ISS for $U = W_0^{1,\frac{4}{3}}(0,\pi)$

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- $a \leq 1 \Rightarrow$ ISS for $U = W_0^{1,\frac{4}{3}}(0,\pi)$

Σ is no more 0-UGASs for $a > 1$.

ISS Lyapunov functions for Sobolev state spaces

$$\frac{\partial x}{\partial t} = c\frac{\partial^2 x}{\partial l^2} + f\big(x(l,t), \tfrac{\partial x}{\partial l}(l,t)\big) + u(l,t), \quad \forall t > 0$$

$$x(0,t) = x(L,t) = 0, \quad \forall t \geq 0.$$

Theorem (A.M., Hiroshi Ito, 2014, submitted to SICON)

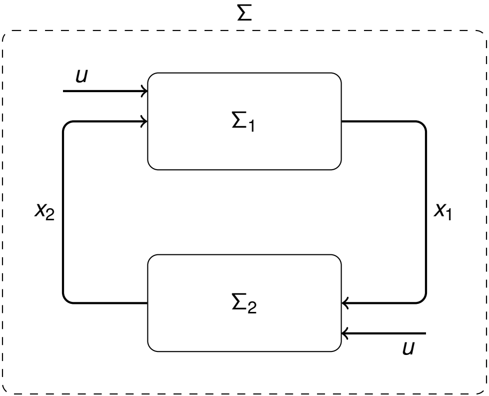
Let $X = W_0^{1,2q}(0,L)$ *and* $\forall x \in X$, *some convex* $\eta : \mathbb{R}_+ \to \mathbb{R}$ *and some* $\epsilon > 0$

$$\int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q-2} \frac{\partial^2 x}{\partial l^2} f\left(x, \tfrac{\partial x}{\partial l}\right) dl \geq \int_0^L \eta\left(\left(\tfrac{\partial x}{\partial l}\right)^{2q}\right) dl$$

$$\hat{\alpha}(s) := \frac{\pi^2}{q^2 L^2}(c-\epsilon)s + L\eta\left(\frac{s}{L}\right) \geq 0, \ \forall s \in \mathbb{R}_+.$$

$$\Rightarrow \quad V(x) = \int_0^L \left(\tfrac{\partial x}{\partial l}\right)^{2q} dl \quad \textit{is ISS-LF} \quad \textit{w.r.t} \quad U = L_{2q}(0,L)$$

Interconnections of 2 systems



$$gain_{\Sigma_1 \to \Sigma_2} \circ gain_{\Sigma_2 \to \Sigma_1}(s) < s \;\Rightarrow \text{iISS of } \Sigma$$

Interconnections of two integral ISS systems

$$\Sigma : \begin{cases} \Sigma_1 : & \dot{x}_1 = A_1 x_1 + f_1(x_1, x_2, u), \ x_1 \in X_1 \\ \Sigma_2 : & \dot{x}_2 = A_2 x_2 + f_2(x_1, x_2, u), \ x_2 \in X_2 \end{cases}$$

ISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is iISS-Lyapunov functions for Σ_i, $i = 1, 2$ iff

- $\dot{V}_1(x_1) \leq -\alpha_1(\|x_1\|_{X_1}) + \sigma_1(\|x_2\|_{X_2}) + \kappa_1(\|u(0)\|_U)$,
- $\dot{V}_2(x_2) \leq -\alpha_2(\|x_2\|_{X_2}) + \sigma_2(\|x_1\|_{X_1}) + \kappa_2(\|u(0)\|_U)$,

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Lyapunov gains

- $gain_{\Sigma_2 \to \Sigma_1} = \alpha_1^{-1} \circ \sigma_1$
- $gain_{\Sigma_1 \to \Sigma_2} = \alpha_2^{-1} \circ \sigma_2$

Small-gain theorem for 2 interconnected iISS systems

Theorem (A. Mironchenko, H. Ito)

Let:

- $\lim_{s\to\infty} \alpha_i(s) = \infty$ *or* $\lim_{s\to\infty} \sigma_{3-i}(s)\kappa_i(1) < \infty$ *for i=1,2.*
- $\exists c > 1$: $\forall s \in \mathbb{R}_+$: $\psi_{11}^{-1} \circ \psi_{12} \circ \alpha_1^{\ominus} \circ c\sigma_1 \circ \psi_{21}^{-1} \circ \psi_{22} \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \leq s$.

$\Rightarrow$ *Σ is iISS.*

If additionally

- $\alpha_i \in \mathcal{K}_\infty$ *for* $i = 1, 2$ $\Rightarrow$ *Σ is ISS.*

iISS (ISS) Lyapunov function:

$$V(x) = \int_0^{V_1(x_1)} \lambda_1(s)ds + \int_0^{V_2(x_2)} \lambda_2(s)ds.$$

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Let $\psi_{11} = \psi_{12} = \psi_1$ and $\psi_{21} = \psi_{22} = \psi_2 \Rightarrow$ SGC:

$$\alpha_1^{\ominus} \circ c\sigma_1 \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \leq s.$$

Example

$$\begin{cases} \dfrac{\partial x_1}{\partial t}(l,t) = \dfrac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\ x_1(0,t) = x_1(\pi,t) = 0; \\ \dfrac{\partial x_2}{\partial t} = \dfrac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2\Big(\dfrac{\partial x_2}{\partial l}\Big)^2 + \Big(\frac{x_1^2}{1+x_1^2}\Big)^{\frac{1}{2}}; \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases}$$

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For what a, b is this system UGASs?

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For what a, b is this system UGASs?

$$X_1 := L_2(0,\pi) \qquad X_2 := H_0^1(0,\pi)$$

Strategy

1. x_1-subsystem is iISS
2. x_2-subsystem is ISS
3. Interconnection is UGASs

$$\begin{cases} \frac{\partial x_1}{\partial t}(l,t) = \frac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\ x_1(0,t) = x_1(\pi,t) = 0; \end{cases}$$

iISS-Lyapunov function:

$$V_1(x_1) := \ln\left(1 + \|x_1\|^2_{L_2(0,\pi)}\right)$$

Lie derivative of V_1:

$$\dot{V}_1 \leq -\frac{2\|x_1\|^2_{L_2(0,\pi)}}{1 + \|x_1\|^2_{L_2(0,\pi)}} + 2\|x_2\|^4_{L_\infty(0,\pi)}.$$

Finally, via Agmon's inequality:

$$\dot{V}_1(x_1) \leq -\underbrace{\frac{2\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}}_{\alpha_1(\|x_1\|_{L_2(0,\pi)})} + \underbrace{8\|x_2\|^4_{H_0^1(0,\pi)}}_{\sigma_1(\|x_2\|_{H_0^1(0,\pi)})}.$$

x_2-subsystem is ISS

$$\begin{cases} \frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial l^2} + a x_2 - b x_2 \left(\frac{\partial x_2}{\partial l}\right)^2 + \overbrace{\left(\frac{x_1^2}{1+x_1^2}\right)^{\frac{1}{2}}}^{u}, \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases}$$

ISS-Lyapunov function:

$$V_2(x_2) = \int_0^\pi \left(\frac{\partial x_2}{\partial l}\right)^2 dl = \|x_2\|^2_{H_0^1(0,\pi)}.$$

Lie derivative of V_2:

$$\dot{V}_2(x_2) \leq -2(1-a-\omega)V_2(x_2) - \frac{2b}{3\pi} V_2^2(x_2) + 2\frac{1}{\omega}\int_0^\pi u^2 dl.$$

$$\dot{V}_2 \leq -2(1-a-\frac{\omega}{2})\|x_2\|^2_{H_0^1(0,\pi)} - \frac{2b}{3\pi}\|x_2\|^4_{H_0^1(0,\pi)} + \frac{\pi}{\omega}\left(\frac{\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}\right).$$

Interconnection is UGASs

Condition for UGASs: for some $c > 0$, for all $s \in \mathbb{R}_+$

$$\underbrace{\psi_{11}^{-1} \circ \psi_{12}}_{=id} \circ \alpha_1^{\ominus} \circ c\sigma_1 \circ \underbrace{\psi_{21}^{-1} \circ \psi_{22}}_{=id} \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \leq s$$

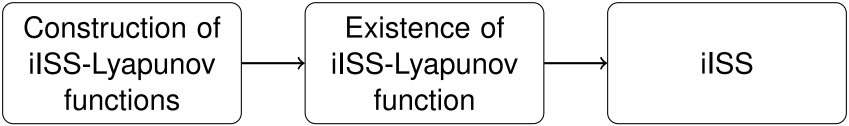
$$\alpha_1(s) = \frac{2s^2}{1+s^2}, \quad \sigma_1(s) = 8s^4, \quad \kappa_1(s) = 0$$

$$\alpha_2(s) = 2\left(1 - a - \frac{\omega}{2}\right)s^2 + \frac{2b}{3\pi}s^4, \; \sigma_2(s) = \frac{\pi}{\omega}\left(\frac{s^2}{1+s^2}\right), \; \kappa_2(s) = 0$$

Condition for UGASs

$$a + \frac{3\pi^2}{b} < 1, \quad b \geq 0.$$

Summary and Outlook



Discussed results

- Bilinear systems
- ISS of parabolic systems over Sobolev spaces
- iISS of parabolic systems over L_q spaces
- ISS/iISS small-gain theorems
- Interconnections of 1-dim parabolic systems

Possible research directions

Linear + Bilinear systems

- Difference between iISS and ISS for linear systems with unbounded B.
- Lyapunov theory for $\dot{x}(t) = Ax(t) + Bu(t)$, with unbounded B.
- Bilinear systems with unbounded bilinear operator.

Characterizations of ISS

- Lyapunov characterization
- Characterization via various robust stability notions

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Thank you for attention!