Dwell-time conditions for robust stability of impulsive systems

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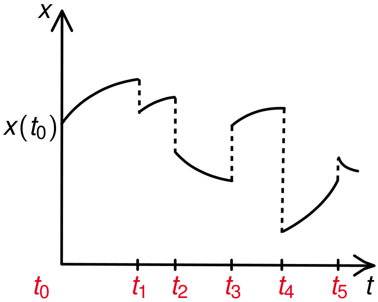
Impulsive systems



$$\dot{x}(t) = f(x(t), u(t)) \quad , \; t \notin \{t_1, t_2, \dots\},$$
$$x(t) = g(x^-(t), u^-(t)) \quad , \; t \in \{t_1, t_2, \dots\}.$$

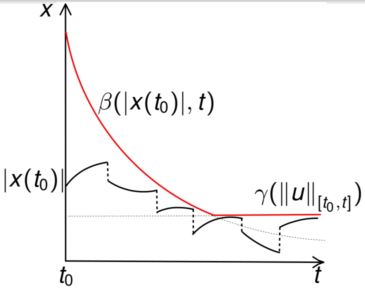
$u \in L_{\infty,\, loc}([t_0, \infty), \mathbb{R}^m)$, $x \in \mathbb{R}^n$ is abs. continuous between impulses,
$x^-(t) := \lim_{s \nearrow t} x(s)$, $u^-(t) := \lim_{s \nearrow t} u(s)$.

Definition (Input-to-state stability (ISS))

Σ is ISS for a given impulse time sequence T, if $\exists \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, s.t.

$$|x(t)| \leq \max\{\beta(|x(t_0)|, t - t_0), \gamma(\|u\|_{[t_0,t]})\}$$

holds for all $x(t_0) \in \mathbb{R}^n$, $u \in L_{\infty,\ loc}([t_0, \infty), \mathbb{R}^m)$, $t \geq t_0$.
Σ is uniformly ISS w.r.t. a class S of impulse time sequences, if it is ISS $\forall T \in S$, and the functions β and γ do not depend on the choice of $T \in S$.



ODE case: $g(x,\cdot) \equiv x$

$$\dot{x}(t) = f(x(t), u(t)) \tag{1}$$

System (1) is ISS, if $\exists \beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ s.t. $\forall\ x(0) \in \mathbb{R}^n$, $u \in L_{\infty,loc}([0,\infty), \mathbb{R}^m)$

$$|x(t)| \leq \max\{\beta(|x(0)|, t), \gamma(\|u\|_{[0,t]})\}, t \geq 0.$$

Definition (ISS-Lyapunov function)

A Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an ISS-LF for (1) if

- $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that $\psi_1(|x|) \leq V(x) \leq \psi_2(|x|)$, $\forall x \in \mathbb{R}^n$
- $\exists \varphi \in \mathcal{P}$ and $\gamma \in \mathcal{K}_\infty$, such that

$$V(x) \geq \gamma(|u|) \Rightarrow \quad \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x))$$

Theorem (Sontag, Wang 1995)

(1) *is ISS* $\Leftrightarrow$ (1) *possesses an ISS-Lyapunov function.*

Exponential ISS-Lyapunov functions

$$\Sigma: \quad \begin{array}{l} \dot{x}(t) = f(x(t), u(t)), \ t \neq t_k, \\ x(t) = g(x^-(t), u^-(t)), \ t = t_k, \ k \in \mathbb{N}. \end{array}$$

Definition

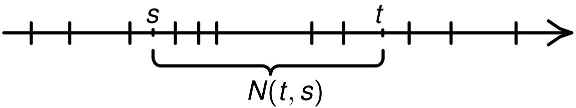
A Lipschitz function $V: \mathbb{R}^n \to \mathbb{R}_+$ is called an exponential ISS-Lyapunov function with rate coefficients c, d for Σ if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), x \in \mathbb{R}^n$$

holds and $\exists c, d \in \mathbb{R}$ and $\gamma \in \mathcal{K}_\infty$ such that

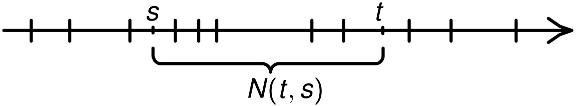
$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -cV(x) \\ V(g(x,u)) \leq e^{-d} V(x). \end{cases}$$

Average dwell time (ADT) condition



$N(t,s)$ is the number of impulse times t_k in $(s,t]$.

Average dwell time (ADT) condition



$N(t,s)$ is the number of impulse times t_k in $(s,t]$.

Theorem (Hespanha, Liberzon, Teel 2008)

Let V be an exponential ISS-LF for Σ with $c, d \in \mathbb{R}$, $d \neq 0$. For arbitrary $\mu, \lambda > 0$, let $\mathcal{S}[\mu,\lambda]$ denote the class of impulse time sequences $\{t_k\}$ satisfying

$$\textit{ADT:} \quad -dN(t,s) - (c-\lambda)(t-s) \leq \mu, \qquad \textit{for any } s,t : 0 \leq s \leq t.$$

Then Σ is uniformly ISS over $\mathcal{S}[\mu,\lambda]$.

Generalized ADT condition

Theorem

Let V be an exponential ISS-Lyapunov function for Σ with corresponding coefficients $c \in \mathbb{R}$, $d \neq 0$.
For arbitrary $h : \mathbb{R}_+ \to (0, \infty)$, for which there exist $g \in \mathcal{L}$: $h(x) \leq g(x)$ for all $x \in \mathbb{R}_+$ consider the class $\mathcal{S}[h]$ of impulse time-sequences, satisfying gADT condition:

$$-dN(t,s) - c(t-s) \leq \ln h(t-s), \quad \forall t \geq s \geq t_0.$$

Then Σ is uniformly ISS over $\mathcal{S}[h]$.

Corollary

Taking $\forall x \in \mathbb{R}_+$ $h(x) = e^{\mu - \lambda x}$, we obtain the ADT condition.

ISS-Lyapunov functions (ISS-LF)

$$\Sigma: \quad \begin{array}{l} \dot{x}(t) = f(x(t), u(t)),\ t \neq t_k, \\ x(t) = g(x^-(t), u^-(t)),\ t = t_k,\ k \in \mathbb{N}. \end{array}$$

Definition (ISS-Lyapunov function)

A Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an ISS-Lyapunov function for Σ if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), x \in \mathbb{R}^n$$

holds and $\exists \gamma \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$, such that

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x)), & \text{f.a.a. } x, u \\ V(g(x,u)) \leq \alpha(V(x)), \ \forall x, u \end{cases}$$

ISS via nonexponential ISS-Lyapunov functions

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x)) \\ V(g(x,u)) \leq \alpha(V(x)). \end{cases}$$

Define $S_\theta := \{\{t_i\}_1^\infty \subset [t_0, \infty) \;:\; t_{i+1} - t_i \geq \theta,\ \forall i \in \mathbb{N}\}$.

Theorem

Let V be an ISS-Lyapunov function for Σ with $\varphi, \alpha \in \mathcal{P}$. Let for some $\theta, \delta > 0$ and all $a > 0$ it hold

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta. \tag{2}$$

Then Σ is ISS for all impulse time sequences $T \in S_\theta$.

ISS via nonexponential ISS-Lyapunov functions-2

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x)) \\ V(g(x,u)) \leq \alpha(V(x)). \end{cases}$$

Define $\tilde{S}_\theta := \{\{t_i\}_1^\infty \subset [t_0, \infty) \ : \ t_{i+1} - t_i \leq \theta, \ \forall i \in \mathbb{N}\}$.

Theorem

Let V be an ISS-Lyapunov function for Σ, α is as in the Definition of ISS-LF and $-\varphi \in \mathcal{P}$. Let for some $\theta, \delta > 0$ and all $a > 0$ it hold

$$\int_{\alpha(a)}^{a} \frac{ds}{\varphi(s)} \geq \theta + \delta. \qquad (3)$$

Then Σ is ISS w.r.t. every sequence from $\tilde{S}_\theta$.

Example

$$\begin{cases} \dot{x} = & -x^3 + u, \ t \notin T \\ x(t) = & x^-(t) + (x^-(t))^3 + u^-(t), \ t \in T. \end{cases} \tag{4}$$

Take $V : \mathbb{R}^+ \to \mathbb{R}^+$ as $V(x) = |x|$.

The Lyapunov gain χ we choose by $\chi(r) = \left(\frac{r}{a}\right)^{\frac{1}{3}}$, $r \in \mathbb{R}^+$, for some $a \in (0,1)$.
Condition $|x| \geq \chi(|u|)$ implies

$$\begin{aligned} \dot{V}(x) &\leq -(1-a)(V(x))^3, \\ V(g(x,u)) &\leq V(x) + (1+a)(V(x))^3. \end{aligned}$$

Integral on the lhs of (2) takes form

$$I(y,a) = \int_y^{y+(1+a)y^3} \frac{dx}{(1-a)x^3} \leq \frac{1+a}{(1-a)}.$$

$\forall \varepsilon > 0$ there exists a_ε such that $I(y, a_\varepsilon) \leq 1 + 2\varepsilon$.

$\forall \varepsilon > 0$ we can choose $\theta := 1 + \varepsilon$. Note, that the smaller θ we take, the larger are the gains.

Summary

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- ISS-Lyapunov functions
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What kind of restrictions do we need?

Dwell-time conditions

generalized ADT

$$-dN(t,s) - c(t-s) \leq \ln h(t-s)$$

fixed DT

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta$$

$h(x) := e^{\mu - \lambda x}$

average DT

$$-dN(t,s) - (c-\lambda)(t-s) \leq \mu$$

$\mu := -d$

$$\frac{1}{\theta} \leq \frac{c-\lambda}{-d}$$

Outlook

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x(t), u(t)) \quad &, \; t \notin \{t_1, t_2, \ldots\}, \\ x(t) &= g(x^-(t), u^-(t)) \quad &, \; t \in \{t_1, t_2, \ldots\}. \end{aligned}$$

$$\begin{aligned} \dot{x}(t) &= f(x_t, u(t)) \quad &, \; t \notin \{t_1, t_2, \ldots\}, \\ x(t) &= g(x_t^-, u^-(t)) \quad &, \; t \in \{t_1, t_2, \ldots\}. \end{aligned}$$

Possible directions of future work

- Interconnections of impulsive ISS systems.
- Dwell-time conditions for hybrid systems.

Thank you for attention!