Constructions of ISS-Lyapunov functions for interconnected impulsive systems

Sergey Dashkovskiy[1] **Andrii Mironchenko**[2]

[1]Department of Civil Engineering
Erfurt University of Applied Sciences

[2]Institute of Mathematics
University of Würzburg

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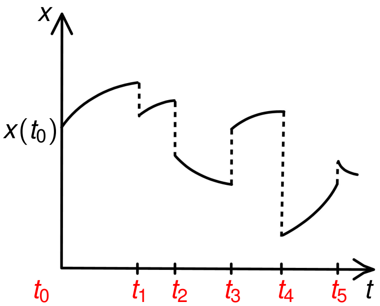


$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \quad , \ t \notin \{t_1, t_2, \dots\}, \\ x(t) &= g(x^-(t), u^-(t)) \quad , \ t \in \{t_1, t_2, \dots\}. \end{aligned}$$

$u \in L_{\infty,\ loc}([t_0, \infty), \mathbb{R}^m)$, $x \in \mathbb{R}^n$ is abs. continuous between impulses,
$x^-(t) := \lim_{s \nearrow t} x(s)$, $u^-(t) := \lim_{s \nearrow t} u(s)$.

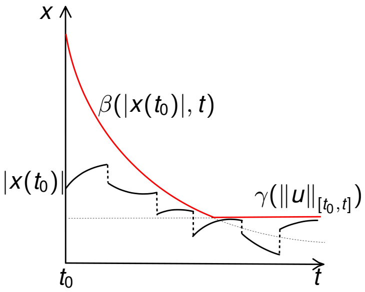
ISS of impulsive system

Definition (Input-to-state stability (ISS))

Σ is ISS for a given impulse time sequence T, if $\exists \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, s.t.

$$|x(t)| \leq \max\{\beta(|x(t_0)|, t - t_0), \gamma(\|u\|_{[t_0,t]})\}$$

holds for all $x(t_0) \in \mathbb{R}^n$, $u \in L_{\infty,\ loc}([t_0, \infty), \mathbb{R}^m)$, $t \geq t_0$.
Σ is uniformly ISS w.r.t. a class S of impulse time sequences, if it is ISS $\forall T \in S$, and the functions β and γ do not depend on the choice of $T \in S$.



ISS-Lyapunov functions (ISS-LF)

$$\Sigma : \quad \begin{array}{l} \dot{x}(t) = f(x(t), u(t)),\ t \neq t_k, \\ x(t) = g(x^-(t), u^-(t)),\ t = t_k,\ k \in \mathbb{N}. \end{array}$$

Definition (ISS-Lyapunov function)

A Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an ISS-Lyapunov function for Σ if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that

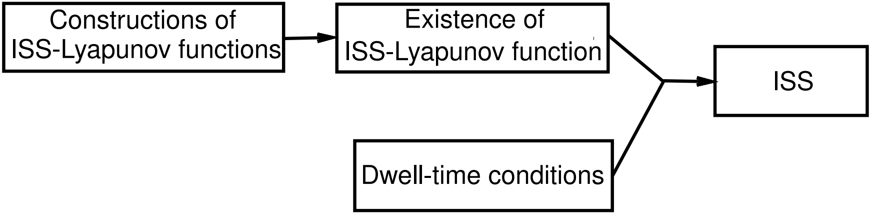
$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), x \in \mathbb{R}^n$$

holds and $\exists \gamma \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$, such that

$$V(x) \geq \gamma(|u|) \Rightarrow \left\{ \begin{array}{ll} \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x)), & \text{f.a.a. } x, u \\ V(g(x,u)) \leq \alpha(V(x)), \ \forall x, u & \end{array} \right.$$

If $\varphi(x) = cx$ and $\alpha(x) = e^{-d}x$, then V is an exponential ISS-Lyapunov function.

Our aims


Constructions of
ISS-Lyapunov functions
Existence of
ISS-Lyapunov function
ISS
Dwell-time conditions

ISS via nonexponential ISS-Lyapunov functions

$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x,u) \leq -\varphi(V(x)) \\ V(g(x,u)) \leq \alpha(V(x)). \end{cases}$$

Define $S_\theta := \{\{t_i\}_1^\infty \subset [t_0, \infty) \ : \ t_{i+1} - t_i \geq \theta, \ \forall i \in \mathbb{N}\}$.

Theorem (S.D., A.M., MTNS 2012)

Let V be an ISS-Lyapunov function for Σ with $\varphi, \alpha \in \mathcal{P}$. Let for some $\theta, \delta > 0$ and all $a > 0$ it hold nonlinear dwell-time condition

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta. \qquad \text{(nDT)}$$

Then Σ is ISS for all impulse time sequences $T \in S_\theta$.

Exponential ISS-Lyapunov functions

$$\Sigma : \quad \begin{array}{l} \dot{x}(t) = f(x(t), u(t)), \ t \neq t_k, \\ x(t) = g(x^-(t), u^-(t)), \ t = t_k, \ k \in \mathbb{N}. \end{array}$$

Definition

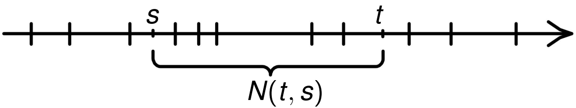
A Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an exponential ISS-Lyapunov function with rate coefficients c, d for Σ if $\exists \ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), x \in \mathbb{R}^n$$

holds and $\exists c, d \in \mathbb{R}$ and $\gamma \in \mathcal{K}_\infty$ such that

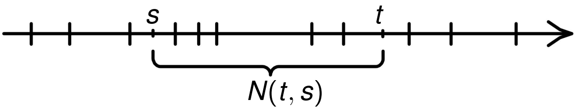
$$V(x) \geq \gamma(|u|) \Rightarrow \begin{cases} \nabla V(x) \cdot f(x, u) \leq -cV(x) \\ V(g(x, u)) \leq e^{-d} V(x). \end{cases}$$

Average dwell time (ADT) condition



$N(t, s)$ is the number of impulse times t_k in $(s, t]$.

Average dwell time (ADT) condition



$N(t,s)$ is the number of impulse times t_k in $(s,t]$.

Theorem (Hespanha, Liberzon, Teel 2008)

Let V be an exponential ISS-LF for Σ with $c, d \in \mathbb{R}$, $d \neq 0$. For arbitrary $\mu, \lambda > 0$, let $\mathcal{S}[\mu,\lambda]$ denote the class of impulse time sequences $\{t_k\}$ satisfying

$$\text{ADT:} \quad -dN(t,s) - (c-\lambda)(t-s) \leq \mu, \quad \text{for any } s,t : 0 \leq s \leq t.$$

Then Σ is uniformly ISS over $\mathcal{S}[\mu,\lambda]$.

Generalized ADT condition

Theorem (S.D., A.M., MTNS 2012)

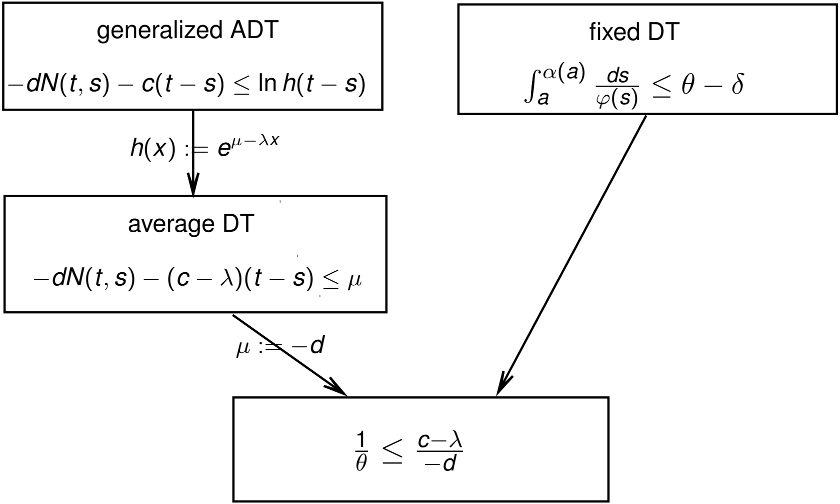
Let V be an exponential ISS-Lyapunov function for Σ with corresponding coefficients $c \in \mathbb{R}$, $d \neq 0$.
For arbitrary $h : \mathbb{R}_+ \to (0, \infty)$, for which there exist $g \in \mathcal{L}$: $h(x) \leq g(x)$ for all $x \in \mathbb{R}_+$ consider the class $\mathcal{S}[h]$ of impulse time-sequences, satisfying gADT condition:

$$-dN(t,s) - c(t-s) \leq \ln h(t-s), \quad \forall t \geq s \geq t_0. \qquad \text{(gADT)}$$

Then Σ is uniformly ISS over $\mathcal{S}[h]$.

Corollary

Taking $\forall x \in \mathbb{R}_+$ $h(x) = e^{\mu - \lambda x}$, we obtain the ADT condition.


generalized ADT
$-dN(t,s) - c(t-s) \le \ln h(t-s)$
fixed DT
$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \le \theta - \delta$
$h(x) := e^{\mu - \lambda x}$
average DT
$-dN(t,s) - (c-\lambda)(t-s) \le \mu$
$\mu := -d$
$\frac{1}{\theta} \le \frac{c-\lambda}{-d}$

ISS-Lyapunov functions for subsystems

$$\Sigma : \quad \begin{cases} \dot{x}_i(t) = f_i(x_1(t), \dots, x_n(t), u(t)), \ t \notin T_i, \\ x_i(t) = g_i(x_1^-(t), \dots, x_n^-(t), u^-(t)), \ t \in T_i, \\ i = \overline{1, n}, \end{cases} \tag{1}$$

ISS-Lyapunov functions for subsystems

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$V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is an ISS-Lyapunov function for i-th subsystem of (1) if:

1. $\exists \psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$: $\quad \psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|), \quad \forall x_i \in \mathbb{R}^{N_i}$
2. There exist $\gamma_{ij}, \gamma_i \in \mathcal{K}$ and $\varphi_i \in \mathcal{P}$, so that
$$V_i(x_i) \geq \max\{\max_{j=1}^{n} \gamma_{ij}(V_j(x_j)), \gamma_i(|\xi|)\} \tag{2}$$
implies
$$\nabla V_i \cdot f_i(x, \xi) \leq -\varphi_i\left(V_i(x_i(t))\right). \tag{3}$$
3. There exist $\alpha_i \in \mathcal{P}$, such that
$$V_i(g_i(x, \xi)) \leq \max\{\alpha_i(V_i(x_i)), \max_{j=1}^{n} \gamma_{ij}(V_j(x_j)), \gamma_i(|\xi|)\}. \tag{4}$$

Small-gain theorem

Let $\Gamma_M = (\gamma_{ij})_{i,j=1,\dots,n}$, $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ (gain matrix).
Let us introduce the gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(s) := \left(\max_{j=1}^{n} \gamma_{1j}(s_j), \dots, \max_{j=1}^{n} \gamma_{nj}(s_j) \right), \; s \in \mathbb{R}^n_+. \qquad (5)$$

Theorem (Small-gain theorem)

Let for every Σ_i there exist an ISS-Lyapunov function V_i with corresponding gains γ_{ij}. If the small-gain condition

$$\Gamma(s) \not\geq s, \; \forall \; s \in \mathbb{R}^n_+ \setminus \{0\},$$

holds, then Σ possesses an ISS-Lyapunov function, defined by

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}. \qquad (6)$$

Idea of the proof. Estimate for continuous flow is as in S. Dashkovskiy, B. Rüffer, F. Wirth (MTNS 2006). To verify estimate of V on the jumps the ideas from D. Nesic, A. Teel (CDC 2008) have be used.

Small-gain theorem for construction of eISS LFs

Let

$$P := \{f : \mathbb{R}_+ \to \mathbb{R}_+ : \exists a \geq 0,\ b > 0 :\ f(s) = as^b\ \forall s \in \mathbb{R}_+\}$$

Theorem (SGT when subsystems have exponential ISS-LF)

Let V_i be eISS Lyapunov function for Σ_i with corresponding gains γ_{ij}, $i = 1, \dots, n$, for which the small-gain condition holds. Let also $\gamma_{ij} \in P$. Then function $V : \mathbb{R}^N \to \mathbb{R}_+$, defined by

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\},$$

is an eISS Lyapunov function for Σ for certain σ.

E.g. $\sigma = Q(at)$, $a > 0$, $Q(x) := MAX\{x, \Gamma(x), \Gamma^2(x), \dots, \Gamma^{n-1}(x)\}$
(due to I. Karafyllis, Z.-P. Jiang (IMA Journal of Math. Cont. and Inf., 2011))

Small-gain contra Dwell-time

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t),\ t \notin T,$$
$$x_1(t) = e^{-1} x_1^-(t),\ t \in T$$

and

$$\dot{x}_2(t) = -x_2(t) + 3\sqrt{|x_1(t)|},\ t \notin T,$$
$$x_2(t) = e^{-1} x_2^-(t),\ t \in T.$$

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$$V_1(x_1) = |x_1|, \quad \gamma_{12}(r) = \frac{1}{a} r^2,$$
$$V_2(x_2) = |x_2|, \quad \gamma_{21}(r) = \frac{1}{b}\sqrt{r},$$

Small-gain contra Dwell-time

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$$x_2(t) = e^{-1} x_2^-(t),\ t \in T.$$

$$V_1(x_1) = |x_1|, \quad \gamma_{12}(r) = \tfrac{1}{a} r^2,$$
$$V_2(x_2) = |x_2|, \quad \gamma_{21}(r) = \tfrac{1}{b} \sqrt{r},$$

$$|x_1| \geq \gamma_{12}(|x_2|) \Rightarrow \dot{V}_1(x_1) \leq (a-1) V_1(x_1),$$
$$|x_2| \geq \gamma_{21}(|x_1|) \Rightarrow \dot{V}_2(x_2) \leq (3b-1) V_2(x_2).$$

Summary

We presented:

- Construction of ISS-Lyapunov function from the ISS-Lyapunov functions
- Construction of exponential ISS-Lyapunov functions, if eISS-Lyapunov functions for subsystems are given and the gains are power functions.

We showed:

- Trade-offs between fulfillment of dwell-time and small-gain conditions

Outlook

Abstract ∞-dim systems

Using framework from

- S. Dashkovskiy, A.M. Input-to-state stability of infinite-dimensional control systems, MCSS, 2013.

one can investigate a system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x(t), u(t)) \quad &, t \notin \{t_1, t_2, \dots\}, \\ x(t) &= g(x^-(t), u^-(t)) \quad &, t \in \{t_1, t_2, \dots\}. \end{aligned}$$

Interconnected impulsive systems with time-delays

$$\begin{aligned} \dot{x}(t) &= f(x_t, u(t)) \quad &, t \notin \{t_1, t_2, \dots\}, \\ x(t) &= g(x_t^-, u^-(t)) \quad &, t \in \{t_1, t_2, \dots\}. \end{aligned}$$

Possible directions of future work

- Interconnections of impulsive ISS systems with different impulse time sequences.
- Trade-offs between the small-gain and dwell-time conditions