

# Stabilization of switched linear differential-algebraic equations via time-dependent switching signals

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- 1 Problem formulation
- 2 Approximation of switched DAEs by switched ODEs
- 3 Conclusions

# Notation for DAEs

Consider a regular DAE

$$E\dot{x} = Ax, \quad (1)$$

## Theorem (Quasi-Weierstraß form)

If (1) is regular, then there exist invertible matrices  $S, T \in \mathbb{R}^{n \times n}$ :

$$SET = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \quad SAT = \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix},$$

where  $N, J$  are square matrices and  $N$  is nilpotent.

$$A^d := T \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \quad \Pi_{(E,A)} := T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^{-1}.$$

The solution of (1) with  $x(0) = x_0 \in \mathcal{C}_{(E,A)}$ , is also a solution of

$$\dot{x} = A^d x.$$

Consider

$$\dot{x} = A_{\sigma(t)}x.$$

$$C_{(I,A_i)} \equiv C_{(I,A_j)} = \mathbb{R}^n$$

Solution of switched ODE:

$$x(t) = e^{A_{r_i}(t-t_i)} \cdot e^{A_{r_{i-1}}(t_i-t_{i-1})} \dots e^{A_{r_1}(t_1-t_0)}x_0,$$

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Let us view (2) as a time-variant DAE

$$E(t)\dot{x} = A(t)x,$$

Solution of (2) (Trenn, 2009):

$$x(t) = e^{A_{r_i}^d(t-t_i)} \Pi_{r_i} \cdot e^{A_{r_{i-1}}^d(t_i-t_{i-1})} \Pi_{r_{i-1}} \cdots e^{A_{r_1}^d(t_1-t_0)} \Pi_{r_1} x_0,$$



$$E(t)\dot{x} = A(t)x, \quad (3)$$

## Definition (Trenn, 2009)

The zero solution is called **globally attractive** if  $\forall x_0$  solutions of (3), does not contain Dirac impulses and its derivatives and converge to zero when  $t \rightarrow \infty$ .

The zero solution is called **globally asymptotically stable (GAS)** if it is globally attractive and Lyapunov stable.

## Definition

The switched system

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)$$

is called **stabilizable via time-dependent switching** if there exists a piecewise continuous switching signal  $\sigma : \mathbb{R}_+ \rightarrow \mathcal{I}$  such that its equilibrium is GAS.

$$\dot{x} = A_i^\varepsilon x$$

with system matrix

$$A_i^\varepsilon := A_i^d \Pi_i - \frac{1}{\varepsilon} (I - \Pi_i) = T_i \begin{pmatrix} J_i & 0 \\ 0 & -\frac{1}{\varepsilon} I \end{pmatrix} T_i^{-1}.$$

For the set of system matrices  $A_i^\varepsilon$  we define the switched linear ODE:

$$\dot{x} = A_{\sigma(t)}^\varepsilon x$$

## Fact

For all  $t > 0$  and  $x_0 \in \mathbb{R}^n$  it holds

$$(e^{A_i^d t} \Pi_i - e^{A_i^\varepsilon t}) x_0 = e^{-\frac{t}{\varepsilon}} (I - \Pi_i) x_0.$$

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)$$

## Theorem (Criterion for stabilizability of a switched DAE)

*The following statements are equivalent:*

- 1 *sDAE is exponentially stabilizable*
- 2 *sDAE is exponentially stabilizable via periodic signal  $\sigma_p$ .*
- 3  $\exists \varepsilon_0 > 0$ : *switched ODE is stabilizable via the same periodic switching signal  $\sigma_p$  for all  $0 < \varepsilon < \varepsilon_0$ , uniformly w.r.t.  $\varepsilon$ :*  
$$\exists \mathbf{s} > \mathbf{0} : \forall \varepsilon \in (0, \varepsilon_0) \Rightarrow |\phi_\varepsilon(\mathbf{s}, x_0, \sigma_p)| \leq \frac{1}{2}|x_0|, \forall x_0 \in \mathbb{R}^n.$$

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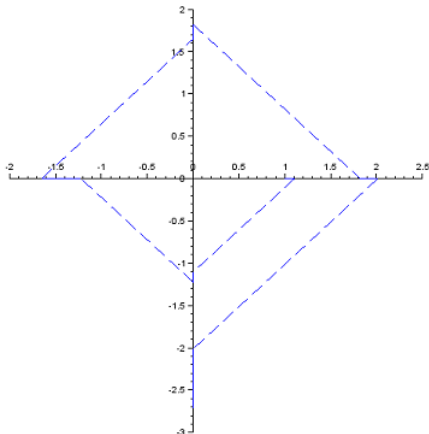
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## Lemma

*Let  $A_i$  be invertible and let  $A_i^d$ ,  $i = 1, \dots, m$  commute pairwise. If (4) is stabilizable for a certain  $\varepsilon_0$ , then it is stabilizable via the same switching signal  $\sigma_p$ , uniformly w.r.t.  $\varepsilon \in (0, \varepsilon_0)$ .*

# Tightness of the above criterion

$$\Pi_1 = A_1^d = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \quad \Pi_2 = A_2^d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$



Now consider the corresponding ODE approximations

$$A_1^\varepsilon = A_1^d \Pi_1 - \frac{1}{\varepsilon}(I - \Pi_1) = \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -1 - \frac{1}{\varepsilon} & 1 \end{pmatrix}$$

and

$$A_2^\varepsilon = A_2^d \Pi_2 - \frac{1}{\varepsilon}(I - \Pi_2) = \begin{pmatrix} 1 & 1 + \frac{1}{\varepsilon} \\ 0 & -\frac{1}{\varepsilon} \end{pmatrix}.$$

Consider the periodic signal  $\sigma_\varepsilon$  of a period  $2\varepsilon$ , defined by

$$\sigma_\varepsilon(t) = \begin{cases} 1, & t \in [0, \varepsilon), \\ 2, & t \in [\varepsilon, 2\varepsilon). \end{cases}$$

$$M_\varepsilon := e^{A_2^\varepsilon} e^{A_1^\varepsilon} = \begin{pmatrix} e^{\varepsilon-1} - (e^\varepsilon - e^{-1})^2 & e^\varepsilon(e^\varepsilon - e^{-1}) \\ -e^{-1}(e^\varepsilon - e^{-1}) & e^{\varepsilon-1} \end{pmatrix}.$$

The state of the system (4) at time  $2\varepsilon k$ , corresponding to  $\sigma_\varepsilon$  is given by

$$\phi_{eps}(2\varepsilon k, x_0) = M_\varepsilon^k x_0.$$

Define  $y(k) := \phi_{eps}(k2\varepsilon, x_0)$ . Then we obtain a discrete system

$$y(k) = M_\varepsilon y(k-1).$$

One can easily compute, that  $\rho(M_0) < 1$ .

## Theorem

*Assume there exist  $\varepsilon_0 > 0$  and  $s > 0$  so that for all  $0 < \varepsilon < \varepsilon_0$  there exists a periodic switching signal  $\sigma_\varepsilon$  with period  $s$  such that*

$$|\phi_\varepsilon(\mathbf{s}, \mathbf{x}_0, \sigma_\varepsilon)| \leq \frac{1}{2} |\mathbf{x}_0|.$$

*Moreover, assume there exists  $t_d > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$ , for any two subsequent switches of  $\sigma_\varepsilon$  it holds  $|t_i^\varepsilon - t_{i+1}^\varepsilon| \geq t_d$ . Then sDAE is stabilizable via  $\sigma_{\varepsilon^*}$  for some  $\varepsilon^* \in (0, \varepsilon_0)$ .*



## We have presented

- Criterion of stabilizability of a switched DAE in terms of ODE approximations

## Further results from the journal version

- DAE generalization of the convex-hull criterion
- Stabilization via consistency projectors.
- Criterion for stabilizability of DAEs with commutative vector fields.

## Possible directions for future work

- Stabilization of DAEs via state-dependent switching
- Connections between state-dependent and time-dependent stabilization.