Input-to-state stability of systems of partial differential equations

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Outline

Semigroups and their Generators

Let X be a Banach space, and $L(X)$ be the space of bounded operators, defined on X.

Definition (Strongly continuous semigroup)

A family of operators $\{T(t),\ t \geq 0\} \subset L(X)$, is called a **strongly continuous semigroup** (for short C_0-semigroup), if it holds

1. $T(0) = I$
2. $T(t+s) = T(t)T(s), \forall t, s \geq 0.$
3. For all $x \in X$ function $t \mapsto T(t)x$ belongs to $C([0,\infty), X)$

Definition (Analytic semigroup)

The C_0-semigroup is called analytic, if in addition it holds:

- $T(t)x \to x$, when $t \to +0$.
- $t \mapsto T(t)x$ is real analytic on $0 < t < \infty$ for every $x \in X$.

Definition (Generator of a C_0-semigroup)

Linear operator L, defined by

$$Lx = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$$

with domain

$$D(L) = \{x \in X : \lim_{t \to +0} \frac{1}{t}(T(t)x - x) \text{ exists}\}$$

is called an infinitesimal generator of a C_0-semigroup $T(t)$.

Definition of control system

Let $(X, \|\cdot\|_X)$ be a state space, $(U, \|\cdot\|_U)$ be an input space and U_c be the set of admissible input functions: $\mathbb{R}_+ \to U$.

Definition (Control system)

The triple $\Sigma = (X, U_c, \phi)$ is a control system, if:

- $\phi(t, t, x, \cdot) = x$ for all $t \geq 0$.
- $\forall\ t \geq r \geq s \geq 0, \forall x \in X, \forall u_1 \in U_c^{[s,r]},\ u_2 \in U_c^{[r,t]}$ it holds $\phi(t, r, \phi(r, s, x, u_1), u_2) = \phi(t, s, x, u)$, where

$$u(\tau) := \begin{cases} u_1(\tau), \tau \in [s, r], \\ u_2(\tau), \tau \in [r, t]. \end{cases}$$

- $\forall x \in X,\ u \in U_c$ the map $t \to \phi(t, 0, x, u)$ is in $C([0, \infty), X)$
- ϕ is continuous in two last arguments.

Stability notions

Let $\Sigma = (X, U_c, \phi)$ be time-invariant and $\phi(t,0,0,0) \equiv 0$.

Definition (Global asymptotic stability at zero)

Σ is *globally asymptotically stable at zero uniformly with respect to x* (0-UGAS*x*), if $\exists \beta \in \mathcal{KL}$: $\forall \phi_0 \in X$, $\forall t \geq 0$ it holds

$$\|\phi(t,0,\phi_0,0)\|_X \leq \beta(\|\phi_0\|_X, t).$$

Definition (Local input-to-state stability)

Σ is *locally input-to-state stable* (LISS), if $\exists \rho_x, \rho_u > 0$ and $\exists \beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, such that $\forall t \geq 0$, $\forall \phi_0 : \|\phi_0\|_X \leq \rho_x$ and $\forall u \in U_c$: $\|u\|_{U_c} \leq \rho_u$ it holds

$$\|\phi(t,t_0,\phi_0,u)\|_X \leq \max\{\beta(\|\phi_0\|_X, t), \gamma(\|u\|_{U_c})\}.$$

Definition (exponential LISS and ISS)

- If $\beta(r,t) = Me^{\omega t}r$, for some $\omega < 0$, then (X, U_c, ϕ) is locally exponentially ISS
- If one can choose $\rho_x = \rho_u = \infty$, then (X, U_c, ϕ) is ISS

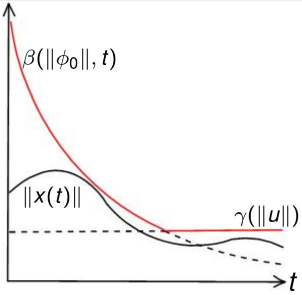

Figure: Input-to-state stability in max-formulation

Outline

LISS-Lyapunov functions

Definition (Local ISS-Lyapunov function (LISS-LF))

A smooth function $V : D \to \mathbb{R}_+$, $D \subset X$, $0 \in int(D)$ is LISS-LF for system (X, U_c, ϕ), if there exist $\rho_x, \rho_u > 0$, functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and positive definite function α, such that:

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in D$$

and $\forall x \in D : \|x\|_X \leq \rho_x$, $\forall u \in U : \|u\|_U \leq \rho_u$ it holds:

$$\|x\|_X \geq \chi(\|u\|_U) \;\Rightarrow\; \dot{V}(x) \leq -\alpha(\|x\|_X), \tag{1}$$

where

$$\dot{V}(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t, 0, x, u)) - V(x)).$$

Function χ is called Lyapunov gain.

Lyapunov characterisation of LISS

Theorem

Let $\Sigma = (X, U_c, \phi)$ be a time-invariant control system. If Σ possesses a LISS-Lyapunov function, then Σ is LISS.

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Example: semilinear heat equation

$$\begin{cases} \frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} - f(s) + u(x,t), \quad x \in (0,\pi),\ t > 0, \\ s(0,t) = s(\pi,t) = 0. \end{cases} \tag{2}$$

We assume, that f is locally Lipschitz, monotonically increasing up to infinity, $f(-r) = -f(r)$ for all $r \in \mathbb{R}$ and $u(\cdot, t) \in L_2(0,\pi)$.

Example: Formulation and Lyapunov function

We define: $As = \frac{d^2 s}{dx^2}$ with $D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$.

Operator A generates an analytic semigroup on $L_2(0,\pi)$.
System (2) takes form

$$\frac{ds}{dt} = As - f(s) + u,\ t > 0. \tag{3}$$

Equation (3) defines the control system with state space $X = H_0^1(0,\pi)$ and input space $U = L_2(0,\pi)$.

The norm on $H_0^1(0,\pi)$ we define as $\|s\|_{H_0^1(0,\pi)} = \left(\int_0^\pi s_x^2(x)dx\right)^{\frac{1}{2}}$.

$$V(s) = \int_0^\pi \left(\frac{1}{2} s_x^2(x) + \int_0^{s(x)} f(y)dy \right) dx. \tag{4}$$

Verification of the first property:

$$\int_0^{s(x)} f(y)dy \geq 0 \quad \Rightarrow \quad V(s) \geq \int_0^{\pi} \frac{1}{2} s_x^2(x)dx = \frac{1}{2}\|s\|^2_{H_0^1(0,\pi)}$$

The derivative of V along the trajectories is:

$$\dot{V}(s) = -\int_0^{\pi} (s_{xx}(x) - f(s(x)))^2 dx + \int_0^{\pi} (s_{xx}(x) - f(s(x)))(-u)dx.$$

Define

$$I(s) := \int_0^{\pi} (s_{xx}(x) - f(s(x)))^2 dx.$$

Using Cauchy-Schwarz inequality for the second term, we have:

$$\dot{V}(s) \leq -I(s) + \sqrt{I(s)}\, \|u\|_{L_2(0,\pi)}. \tag{5}$$

Lyapunov function

$$I(s) := \int_0^{\pi} (s_{xx}(x) - f(s(x)))^2 dx.$$

One can prove directly:

$$I(s) \geq \int_0^{\pi} s_{xx}^2(x) dx.$$

For $s \in H_0^1(0,\pi) \cap H^2(0,\pi)$ it holds (a corollary of Friedrich's inequality), that:

$$\int_0^{\pi} s_{xx}^2(x) dx \geq \int_0^{\pi} s_x^2(x) dx.$$

Overall, we have:

$$I(s) \geq \|s\|^2_{H_0^1(0,\pi)}. \tag{6}$$

Gains

Now we choose the gain as

$$\chi(r) = ar, \ a > 1.$$

If $\|s\|_{H_0^1(0,\pi)} \geq \chi(\|u\|_{L_2(0,\pi)})$, we obtain

$$\dot{V}(s) \leq -I(s) + \frac{1}{a}\sqrt{I(s)}\|s\|_{H_0^1(0,\pi)} \leq -(1-\frac{1}{a})I(s) \leq -(1-\frac{1}{a})\|s\|^2_{H_0^1(0,\pi)}.$$

This proves, that V is ISS-Lyapunov function, and consequently, our control system (with $X = H_0^1(0,\pi)$, $U = L_2(0,\pi)$) is ISS.

Outline

Linear systems

Let A be a generator of a C_0-semigroup $T(t)$ and $B \in L(U,X)$.

$$\begin{aligned} \dot{s} &= As + Bu, \\ s(0) &= s_0. \end{aligned} \tag{7}$$

$$U_c = \{g : \mathbb{R}_+ \to U : \ g \text{ is locally Hölder continuous}\}.$$

$$s(t) = T(t)s_0 + \int_0^t T(t-r)Bu(r)dr. \tag{8}$$

Let $\exists \omega_0 : \forall \omega > \omega_0 \ \exists M_\omega$, such that $\|T(t)\| \leq M_\omega e^{\omega t}$.

$$\|s(t)\|_X \leq M_\omega e^{\omega t} \|s_0\|_X + \frac{M_\omega}{|\omega|} \|B\| \|u\|_{U_c}.$$

Linearisation

Let X be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, and A generates an analytic semigroup on X. Consider a system

$$\dot{x} = Ax + f(x, u), \;\; x(t) \in X, u(t) \in U. \tag{9}$$

Theorem (Linearisation theorem)

Let for some $B \in L(X)$ and $C \in L(U, X)$ it holds

$$f(x, u) = Bx + Cu + g(x, u).$$

Let $\forall w > 0 \; \exists \rho > 0$, s.t. $\forall x, u : \|x\|_X \leq \rho, \;\; \|u\|_U \leq \rho$ it holds

$$\|g(x, u)\|_X \leq w(\|x\|_X + \|u\|_U).$$

If the system

$$\dot{x} = Ax + Bx + Cu \tag{10}$$

is exponentially ISS, then (9) is LISS.

Idea of the proof of linearisation theorem

- $R = A + B$ generates an exponentially stable analytic semigroup.
- $V(x) = \langle Px, x \rangle$ is Lyapunov function for linear system, where $P \in L(X)$ is positive, and

$$\langle Rx, Px \rangle + \langle Px, Rx \rangle = -\|x\|_X^2, \quad \forall x \in D(R).$$

- Assuming $x \in D(R) \subset X$, we compute derivative of V with respect to nonlinear system:

$$\dot{V}(x) \leq -(1 - 2w\|P\|)\|x\|_X^2 + 2\|P\|(\|C\| + w)\|x\|_X\|u\|_U$$

- Take $\chi(r) := \sqrt{r}$. For $\|x\|_X \geq \chi(\|u\|_U)$ we have:

$$\dot{V}(x) \leq -(1 - 2w\|P\|)\|x\|_X^2 + 2\|P\|(\|C\| + w)\|x\|_X^3. \quad (11)$$

- Prove the same for $x \in X \backslash D(R)$.

Outline

Monotonicity of control systems

Definition (Positive cone)

Let X be Banach. A set $K \subset X$ is called a positive cone in X, if:
$\forall a \in \mathbb{R}_+ \; aK \subset K, \quad K + K \subset K, \quad K \cap (-K) = \{0\}$

Definition (Ordered Banach space)

A pair (X, K), where X is Banach, and $K \subset X$ is a positive cone, is called an ordered Banach space with an order $\leq$, given by $x \leq y \; \Leftrightarrow \; y - x \in K$.

Definition (Monotonicity)

Control system $S = (X, U_c, \phi)$ is called monotone, if $\forall t_0 \in \mathbb{R}_+$, for all $t \geq t_0$, $u_1, u_2 \in U_c : u_1 \leq u_2$, $\forall \phi_1, \phi_2 \in X : \phi_1 \leq \phi_2$ it holds $\phi(t, t_0, \phi_1, u_1) \leq \phi(t, t_0, \phi_2, u_2)$.

ISS of Monotone Systems

$$\begin{cases} \frac{\partial s(x,t)}{\partial t} = c^2 \Delta s + f(s, u(x,t)), \ x \in G \subset \mathbb{R}^p, \ t > 0, \\ s(x,0) = \phi_0(x), x \in G, \\ \frac{\partial s}{\partial n}\Big|_{\partial G \times \mathbb{R}_{\geq 0}} = 0. \end{cases} \quad (12)$$

Assume that $f(0,0) = 0 \ \forall t \geq 0$.

Theorem

Let system (12) be monotone. If the system

$$\begin{cases} \frac{ds(x,t)}{dt} = f(s,u), \ x \in G, \ t > 0, \\ s(0) = \phi_0 \end{cases}$$

is ISS, then (12) is also ISS.

Outline

Interconnections of control systems

Let X_i, $i=1,\ldots,n$ be Banach and A_i generate C_0-semigroup on X_i.

$$\Sigma : \begin{cases} \Sigma_i : \quad \dot{x}_i = A_i x_i + f_i(x_1,\ldots,x_n,u), \;\; x_i \in X_i \\ i=1,\ldots,n \end{cases} \tag{13}$$

$X = X_1 \times \ldots \times X_n$ is Banach with $\|\cdot\|_X := \|\cdot\|_{X_1} + \ldots + \|\cdot\|_{X_n}$.

$$A = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_n \end{pmatrix} \qquad D(A) = D(A_1) \times \ldots \times D(A_n).$$

We rewrite the system (13) in vector form:

$$\Sigma : \quad \dot{x} = Ax + f(x,u) \tag{14}$$

$$\begin{cases} \dot{x}_i = A_i x_i + f_i(x_1, \dots, x_n, u), \;\; x_i \in X_i \\ i = 1, \dots, n \end{cases} \qquad (15)$$

ISS-LF for *i*-th subsystem

A smooth function $V_i : X_i \to \mathbb{R}_+$, is a ISS-Lyapunov function (ISS-LF) for i-th subsystem of (13), if $\exists \psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, $\chi_{ij}, \chi_i \in \mathcal{K}$, $j = 1, \dots, n$, $\chi_{ii} := 0$ and positive definite function α_i, such that:

$$\psi_{i1}(\|x_i\|_{X_i}) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|_{X_i}), \quad \forall x_i \in X_i,$$

and $\forall x_i \in X_i$ it holds:

$$V_i(x_i) \geq \max\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|u\|_U)\} \;\Rightarrow\; \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)),$$

$$\dot{V}(x_i) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi_i(t, 0, x_i, u))) - V(x_i)).$$

Small-gain theorem

Let $\Gamma_M = (\chi_{ij})_{i,j=1,\ldots,n}$, $\chi_{ij} \in \mathcal{K}_\infty \cup 0$ (gain matrix).
Let us introduce the gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(s) := \left(\max_{j=1}^{n} \chi_{1j}(s_j), \ldots, \max_{j=1}^{n} \chi_{nj}(s_j) \right), \; s \in \mathbb{R}^n_+. \tag{16}$$

Theorem (Small-gain theorem)

Let for all Σ_i there exist ISS-Lyapunov function V_i with corresponding gains χ_{ij}. If $\Gamma(s) \not\geq s$, $\forall\, s \in \mathbb{R}^n_+ \setminus \{0\}$, then Σ is ISS and possesses ISS-Lyapunov function, defined by

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}. \tag{17}$$

Construction of the path σ

The path σ can be constructed as follows:

$$\sigma(t) = Q(at),\ \forall t \in [0,\infty),\ \forall a > 0,$$

where

$$Q(x) = MAX\{x, \Gamma(x), \Gamma^2(x), \ldots, \Gamma^{n-1}(x)\},$$

and MAX for all $u_i \in \mathbb{R}^n$, $i = 1, \ldots, m$ is defined as

$$z = MAX\{u_1, \ldots, u_m\} \in \mathbb{R}^n, \quad z_i = \max\{u_{1i}, \ldots, u_{mi}\}.$$

The Lyapunov gain of the whole system is

$$\chi(r) := \max_i \sigma_i^{-1}(\chi_i(r)).$$

Conclusions

Main results

- Verification of (L)ISS of distributed parameter systems
  1. LISS-Lyapunov functions
  2. Linearisation
  3. Monotone control systems
  4. Small-gain theorems

Possible directions of future work

- Generalization of the known results from finite-dimensional theory.
- Definitions of stability for the inifinite-dimensional systems.
- Application of the obtained results.

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Thank you for attention!