Construction of iISS Lyapunov functions for interconnected parabolic systems

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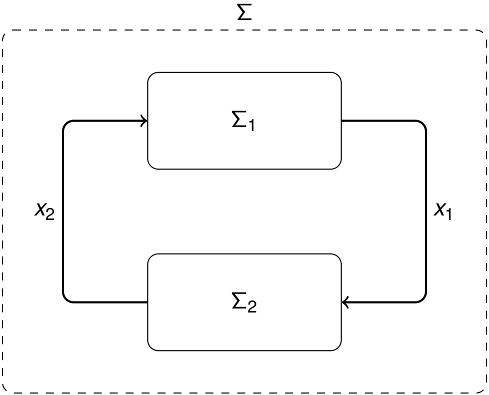
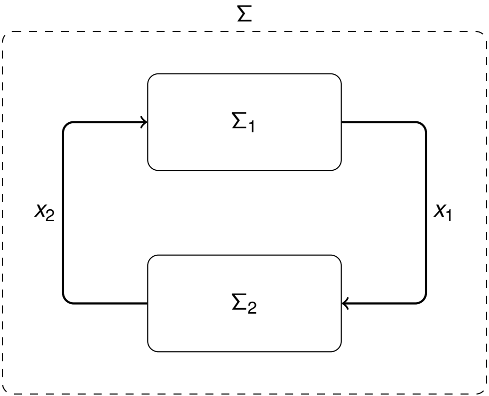


Σ
Σ1
x2
x1
Σ2



- If Σ_i are ODEs $\Rightarrow$ Theory is well-developed.
  - Jiang, Mareels, Wang. Automatica, 1996
  - Dashkovskiy, Rüffer, Wirth. SICON, 2010
  - Dashkovskiy, Ito, Wirth. Eur. Jour. of Cont., 2011
- Our interest: Σ_i - nonlinear parabolic PDEs.

Class of systems

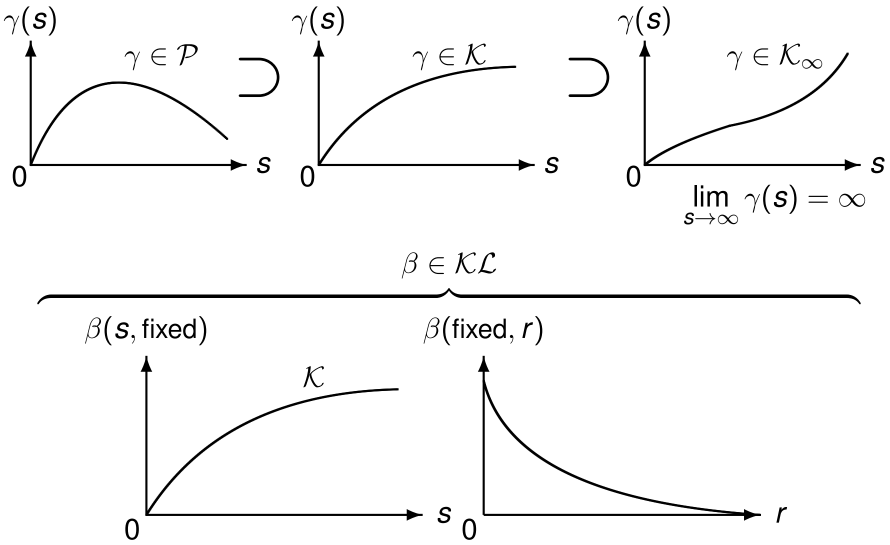
$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), & x(t) \in D(A) \subset X, \\ x(0) = \phi_0. \end{cases}$$

- X = State space
- $\mathcal{U} = PC(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.

$x \in C([0, T], X)$ is a mild solution iff

$$x(t) = T(t)\phi_0 + \int_0^t T(t-s)f(x(s), u(s))ds.$$

Comparison functions



(Integral) input-to-state stability

Definition (GAS uniform w.r.t. state (0-UGASs))

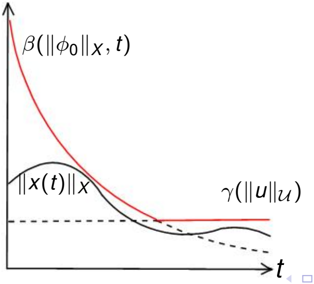
0-UGASs $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}: \quad \forall \phi_0 \in X, \forall t \geq 0$

$$\|\phi(t,\phi_0,0)\|_X \leq \beta(\|\phi_0\|_X, t).$$

Definition (ISS)

ISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}: \quad \forall t \geq 0, \forall \phi_0 \in X, \forall u \in \mathcal{U}$

$$\|\phi(t,\phi_0,u)\|_X \leq \max\Big\{\beta(\|\phi_0\|_X, t), \underbrace{\gamma}_{Gain}\big(\sup_{s\in[0,t]} \|u(s)\|_U\big)\Big\}.$$



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- S. Dashkovskiy, A. M. Input-to-state stability of infinite-dimensional control systems, MCSS, 2013.
- F. Mazenc, C. Prieur. Strict Lyapunov functions for semilinear parabolic partial differential equations, MCRF, 2011.

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Definition (integral input-to-state stability (iISS))

iISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \alpha, \mu \in \mathcal{K}: \quad \forall t \geq 0, \forall \phi_0 \in X, \forall u \in \mathcal{U}$

$$\alpha(\|\phi(t,\phi_0,u)\|_X) \leq \beta(\|\phi_0\|_X, t) + \int_0^t \mu(\|u(s)\|_U)ds.$$

Problem statement

Aim: Efficient framework for study of global stability of interconnected nonlinear parabolic systems

State of the art

1. ISS theory of parabolic systems is developed in L_p setting
2. Many real-world systems are not ISS but are iISS
3. iISS framework is more general and more flexible than ISS

Challenges

1. L_p setting is too restrictive for iISS theory
2. Sobolev spaces should be used as state or input space
3. New constructions of ISS/iISS Lyapunov function are needed

Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),$$
$$x(0) = \phi_0,$$

Definition

$V : X \to \mathbb{R}_+$ is an iISS-Lyapunov function iff $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$ and $\sigma, \alpha \in \mathcal{K}$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $\dot{V}_u(x) \leq -\alpha(V(x)) + \sigma(\|u(0)\|_U)$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x)).$$

$\alpha \in \mathcal{K}_\infty \Rightarrow V$ is an ISS-Lyapunov function.

Theorem

$\exists$ *ISS/iISS Lyapunov function* $\Rightarrow$ *ISS/iISS.*

Stability for interconnected iISS systems

Decompose Σ into Σ_1, Σ_2

↓

Choose state spaces for Σ_1, Σ_2

↓

Find ISS/iISS Lyapunov functions for Σ_1, Σ_2

↓

Apply iISS small-gain theorem

↓

iISS

Outline

Stability for interconnected iISS systems

Decompose Σ into Σ_1, Σ_2

↓

Choose state spaces for Σ_1, Σ_2

↓

Find ISS/iISS Lyapunov functions for Σ_1, Σ_2

↓

Apply iISS small-gain theorem

↓

iISS

ISS Lyapunov functions for Sobolev state spaces

$$\frac{\partial x}{\partial t} = c\frac{\partial^2 x}{\partial l^2} + f\big(x(l,t), \tfrac{\partial x}{\partial l}(l,t)\big) + u(l,t), \quad \forall t > 0$$
$$x(0,t) = x(L,t) = 0, \quad \forall t \geq 0.$$

Theorem (A.M., Hiroshi Ito)

Let $X = H_0^1(0,L)$ *and for some convex* $\eta : \mathbb{R}_+ \to \mathbb{R}$ *and* $\epsilon > 0$

$$\int_0^L \frac{\partial^2 x}{\partial l^2} f\left(x, \tfrac{\partial x}{\partial l}\right) dl \geq \int_0^L \eta\left(\left(\tfrac{\partial x}{\partial l}\right)^2\right) dl$$

$$\hat{\alpha}(s) := \frac{\pi^2}{L^2}(c-\epsilon)s + L\eta\left(\frac{s}{L}\right) \geq 0, \ \forall s \in \mathbb{R}_+.$$

$\Rightarrow \quad V(x) = \int_0^L \left(\frac{\partial x}{\partial l}\right)^2 dl \quad$ *is ISS-LF* $\quad$ *w.r.t* $\quad U = L_2(0,L)$ □

Is applicable e.g. for polynomial nonlinearities

iISS Lyapunov functions in $L_2(0, L)$

$$\frac{\partial x}{\partial t} = c\frac{\partial^2 x}{\partial l^2} + f(x(l,t), u(l,t)), \quad \forall t > 0$$

$$x(0,t)\frac{\partial x}{\partial l}(0,t) = x(L,t)\frac{\partial x}{\partial l}(L,t) = 0, \quad \forall t \geq 0$$

Theorem (A.M., Hiroshi Ito)

Let $X = L_2(0, L)$ and let $\exists \alpha \in \mathcal{K}_\infty \cup \{0\}$, $\sigma \in \mathcal{K}$: $W : y \mapsto y^2$ satisfies

$$\dot{W}(y) := 2yf(y,u) \leq -\alpha(W(y)) + W(y)\sigma(|u|)$$

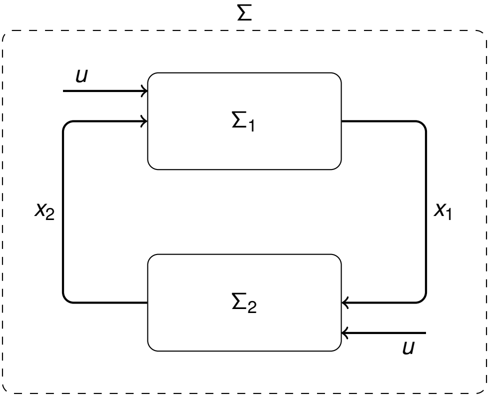
Let any of the following conditions hold:

1. $x(0,t) = 0$ *for all* $t \geq 0$ *or* $x(L,t) = 0$ *for all* $t \geq 0$.
2. α *is convex and* $\mathcal{K}_\infty$.

$\Rightarrow$ $V(x) = \ln(1 + \|x\|^2_{L_2(0,L)})$ *is iISS-LF w.r.t.* $U = H^1_0(0, L)$. □

Is applicable e.g. for bilinear systems

Interconnections of 2 iISS systems



$$gain_{\Sigma_1 \to \Sigma_2} \circ gain_{\Sigma_2 \to \Sigma_1}(s) < s \ \Rightarrow \text{iISS of } \Sigma$$

Interconnections of two iISS systems

$$\Sigma : \begin{cases} \Sigma_1 : \quad \dot{x}_1 = A_1 x_1 + f_1(x_1, x_2, u), \;\; x_1 \in X_1 \\ \Sigma_2 : \quad \dot{x}_2 = A_2 x_2 + f_2(x_1, x_2, u), \;\; x_2 \in X_2 \end{cases}$$

iISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is iISS-Lyapunov functions for Σ_i, $i = 1, 2$ iff

- $\dot{V}_1(x_1) \leq -\alpha_1(\|x_1\|_{X_1}) + \sigma_1(\|x_2\|_{X_2}) + \kappa_1(\|u(0)\|_U)$,
- $\dot{V}_2(x_2) \leq -\alpha_2(\|x_2\|_{X_2}) + \sigma_2(\|x_1\|_{X_1}) + \kappa_2(\|u(0)\|_U)$,

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Lyapunov gains

- $gain_{\Sigma_2 \to \Sigma_1} := \alpha_1^{\ominus} \circ \sigma_1$
- $gain_{\Sigma_1 \to \Sigma_2} := \alpha_2^{\ominus} \circ \sigma_2$

$$\omega^{\ominus}(s) := \begin{cases} \omega^{-1}(s) & \text{, if } s \in \operatorname{Im} \omega \\ +\infty & \text{, otherwise} \end{cases}$$

Small-gain theorem for 2 interconnected iISS systems

Theorem (A. Mironchenko, H. Ito)

Let:

- $V_i(x_i) = \psi_i(\|x_i\|_{X_i})$
- $\alpha_i \in \mathcal{K}_\infty$ *or* $\lim_{s\to\infty}\sigma_{3-i}(s)\kappa_i(1) < \infty$ *for i=1,2.*
- $\exists c > 1$*:* $\forall s \in \mathbb{R}_+$*:*

$$\underbrace{\alpha_1^{\ominus} \circ c\sigma_1}_{\approx gain_{\Sigma_2\to\Sigma_1}} \circ \underbrace{\alpha_2^{\ominus} \circ c\sigma_2}_{\approx gain_{\Sigma_1\to\Sigma_2}} (s) \le s.$$

$\Rightarrow$ *Σ is iISS.*

If additionally

- $\alpha_i \in \mathcal{K}_\infty$ *for* $i = 1, 2$ $\Rightarrow$ *Σ is ISS.*

iISS-LF: $$V(x) = \int_0^{V_1(x_1)} \lambda_1(s)ds + \int_0^{V_2(x_2)} \lambda_2(s)ds.$$

Example

$$\begin{cases} \dfrac{\partial x_1}{\partial t}(l,t) = \dfrac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t)x_2^4(l,t), \\ x_1(0,t) = x_1(\pi,t) = 0; \\ \dfrac{\partial x_2}{\partial t} = \dfrac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2\left(\dfrac{\partial x_2}{\partial l}\right)^2 + \left(\frac{x_1^2}{1+x_1^2}\right)^{\frac{1}{2}}, \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases}$$

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For what a, b is this system UGASs?

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For what a, b is this system UGASs?

$$X_1 := L_2(0,\pi) \qquad X_2 := H_0^1(0,\pi)$$

Strategy

1. x_1-subsystem is iISS
2. x_2-subsystem is ISS
3. Interconnection is UGASs

- iISS-LF for Σ_1: $V_1(x_1) := \ln\left(1 + \|x_1\|^2_{L_2(0,\pi)}\right)$
- Lie derivative of V_1: $\dot{V}_1(x_1) \leq -\underbrace{\frac{2\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}}_{\alpha_1(\|x_1\|_{L_2(0,\pi)})} + \underbrace{8\|x_2\|^4_{H^1_0(0,\pi)}}_{\sigma_1(\|x_2\|_{H^1_0(0,\pi)})}$

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- ISS-LF for Σ_2: $V_2(x_2) = \int_0^\pi \left(\frac{\partial x_2}{\partial l}\right)^2 dl = \|x_2\|^2_{H^1_0(0,\pi)}$
- Lie derivative of V_2:

$$\dot{V}_2 \leq -\underbrace{2\left(1 - a - \frac{\omega}{2}\right)\|x_2\|^2_{H^1_0(0,\pi)} - \frac{2b}{3\pi}\|x_2\|^4_{H^1_0(0,\pi)}}_{\alpha_2(\|x_2\|_{H^1_0(0,\pi)})} + \underbrace{\frac{\pi}{\omega}\left(\frac{\|x_1\|^2_{L_2(0,\pi)}}{1 + \|x_1\|^2_{L_2(0,\pi)}}\right)}_{\sigma_2(\|x_1\|_{L_2(0,\pi)})}$$

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Condition for UGASs: for some $c > 0$, for all $s \in \mathbb{R}_+$

$$\alpha_1^{\ominus} \circ c\sigma_1 \circ \alpha_2^{\ominus} \circ c\sigma_2(s) \leq s$$

- iISS-LF for Σ_1: $V_1(x_1) := \ln\left(1 + \|x_1\|^2_{L_2(0,\pi)}\right)$
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$$\dot{V}_2 \leq \underbrace{-2\left(1 - a - \tfrac{\omega}{2}\right)\|x_2\|^2_{H^1_0(0,\pi)} - \frac{2b}{3\pi}\|x_2\|^4_{H^1_0(0,\pi)}}_{\alpha_2(\|x_2\|_{H^1_0(0,\pi)})} + \underbrace{\frac{\pi}{\omega}\left(\frac{\|x_1\|^2_{L_2(0,\pi)}}{1 + \|x_1\|^2_{L_2(0,\pi)}}\right)}_{\sigma_2(\|x_1\|_{L_2(0,\pi)})}$$

Condition for UGASs

$$a + \frac{3\pi^2}{b} < 1, \quad b > 0.$$

Summary and Outlook

iISS framework for interconnections of 1-dim parabolic systems

- ISS of parabolic systems over Sobolev spaces
- iISS of parabolic systems over L_q spaces
- iISS small-gain theorems

Journal version

A.Mironchenko, H. Ito. *Construction of Lyapunov functions for interconnected parabolic systems: an iISS approach.* Accepted to SICON.

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