

Lyapunov methods for nonlinear integral input-to-state stable systems

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ISS Day in Bergische Universität Wuppertal
16 Februar 2014

- 1 Basic definitions
- 2 Bilinear systems
- 3 ISS and iISS of single nonlinear systems
- 4 Interconnections of iISS systems
- 5 Summary and Outlook

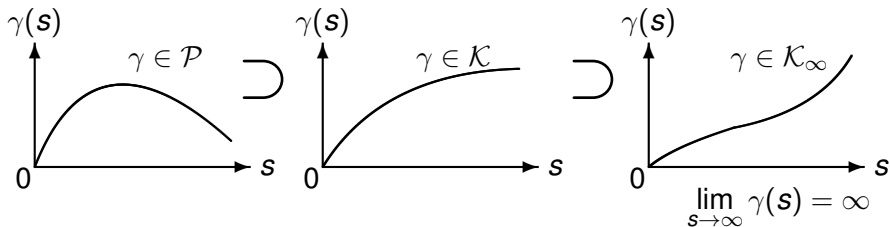
$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), & x(t) \in D(A) \subset X, \\ x(0) = \phi_0. \end{cases}$$

- $U_c = C(\mathbb{R}_+, U)$
- $Ax = \lim_{t \rightarrow +0} \frac{1}{t}(T(t)x - x)$.
- $f(0, 0) = 0$.

$x \in C([0, T], X)$ is a **mild solution** iff

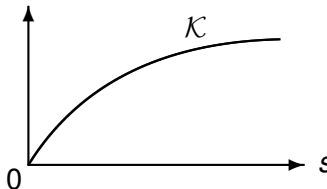
$$x(t) = T(t)\phi_0 + \int_0^t T(t-s)f(x(s), u(s))ds.$$

Comparison functions

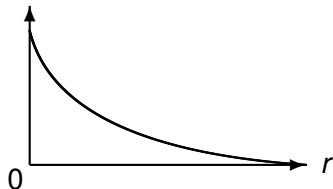


$\beta \in \mathcal{KL}$

$\beta(s, \cdot)$



$\beta(\cdot, r)$



Input-to-state stability

Definition (GAS uniform w.r.t. state (0-UGASS))

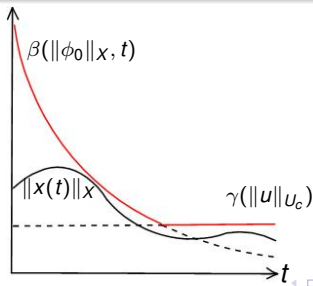
0-UGASS $:\Leftrightarrow \exists \beta \in \mathcal{KL}: \forall \phi_0 \in X, \forall t \geq 0$

$$\|\phi(t, \phi_0, \mathbf{0})\|_X \leq \beta(\|\phi_0\|_X, t).$$

Definition (ISS)

ISS $:\Leftrightarrow \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty: \forall t \geq 0, \forall \phi_0 \in X, \forall u \in U_c$

$$\|\phi(t, \phi_0, u)\|_X \leq \max \left\{ \beta(\|\phi_0\|_X, t), \underbrace{\gamma}_{\text{Gain}} \left(\sup_{s \in [0, t]} \|u(s)\|_U \right) \right\}.$$



Input-to-state stability

Definition (GAS uniform w.r.t. state (0-UGASs))

$$\text{0-UGASs} \quad :\Leftrightarrow \quad \exists \beta \in \mathcal{KL}: \quad \forall \phi_0 \in X, \forall t \geq 0 \\ \|\phi(t, \phi_0, \mathbf{0})\|_X \leq \beta(\|\phi_0\|_X, t).$$

Definition (ISS)

$$\text{ISS} \quad :\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall \phi_0 \in X, \forall u \in U_c \\ \|\phi(t, \phi_0, u)\|_X \leq \max \left\{ \beta(\|\phi_0\|_X, t), \underbrace{\gamma}_{\text{Gain}} \left(\sup_{s \in [0, t]} \|u(s)\|_U \right) \right\}.$$

Definition (integral input-to-state stability (iISS))

$$\text{iISS} \quad :\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \alpha, \mu \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall \phi_0 \in X, \forall u \in U_c \\ \alpha(\|\phi(t, \phi_0, u)\|_X) \leq \beta(\|\phi_0\|_X, t) + \int_0^t \mu(\|u(s)\|_U) ds.$$

Applications of ISS/iISS

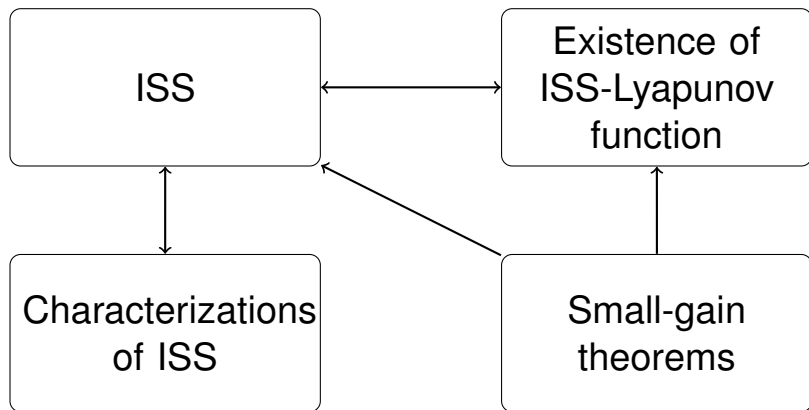
- 1 Unified theory of internal and external stability
- 2 Robust stabilization of nonlinear systems
- 3 Design of robust nonlinear observers
- 4 Nonlinear detectability
- 5 ISS feedback redesign
- 6 Stability of nonlinear networked control systems
- 7 ...

Applications of ISS/iISS

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ISS/iISS framework has been developed for

- 1 ODEs
- 2 Time-delay systems
- 3 Discrete-time systems
- 4 Hybrid and impulsive systems



$$\begin{aligned}\dot{x}(t) &= Ax(t) + f(x(t), u(t)), \\ x(0) &= \phi_0,\end{aligned}$$

Definition

$V : X \rightarrow \mathbb{R}_+$ is an **iISS-Lyapunov function** iff $\exists \psi_1, \psi_2, \sigma, \alpha \in \mathcal{K}$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $\dot{V}_u(x) \leq -\alpha(V(x)) + \sigma(\|u(0)\|_U),$

$$\dot{V}_u(x) = \overline{\lim}_{t \rightarrow +0} \frac{1}{t} (V(\phi(t, x, u)) - V(x)).$$

If $\alpha \in \mathcal{K}_\infty$, then V is called an **ISS-Lyapunov function**.

Theorem

\exists *ISS/iISS Lyapunov function* \Rightarrow *ISS/iISS*.

Definition

V is ISS-Lyapunov function **in implicative form** iff $\exists \psi_1, \psi_2, \gamma, \alpha \in \mathcal{K}_\infty$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $V(x) \geq \gamma(\|u(0)\|_U) \Rightarrow \dot{V}_u(x) \leq -\alpha(V(x)).$

Definition

V is ISS-Lyapunov function **in implicative form** iff $\exists \psi_1, \psi_2, \gamma, \alpha \in \mathcal{K}_\infty$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $V(x) \geq \gamma(\|u(0)\|_U) \Rightarrow \dot{V}_u(x) \leq -\alpha(V(x)).$

Theorem

- *If V is a dissipative ISS-LF $\Rightarrow V$ is an implicative ISS-LF*
- *Let V be an implicative ISS-LF and*
 - 1 *f be Lipschitz on bounded subsets of X*
 - 2 *A generate an analytic semigroup.*
 - 3 *Fréchet derivative $\frac{\partial V}{\partial x}$ exist and be bounded on bounded balls.*
 - 4 *$\exists p \in \mathcal{K}, q \in \mathcal{K}_\infty: \|\frac{\partial V}{\partial x} f(0, v)\|_X \leq p(\|x\|_X) + q(\|v\|_U)$*

Then some dissipative ISS-LF W exists.

How tight is this theorem? \Rightarrow ???

Strategy

- Choose a Lyapunov function V
- Compute the derivative of V (possibly on the dense subset of X).
- Verify the dissipative inequality for \dot{V} on a dense subset of X .
- Use density argument!
 - 1 Using smoothification (if A generates an analytic semigroup)
 - 2 Using continuous dependence on initial data (general case)

$$\dot{x}(t) = Ax(t) + Bu(t) + C(x(t), u(t)),$$

where $B \in L(U, X)$, and $\exists K > 0$:

$$\|C(x, u)\|_X \leq K\|x\|_X\|u\|_U.$$

The simplest example:

$$\dot{x} = -x + xu.$$

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The simplest example:

$$\dot{x} = -x + xu.$$

For $V(x) = \ln(1 + x^2)$ we have

$$\dot{V}(x) = \frac{2x\dot{x}}{1+x^2} = -\frac{2x^2}{1+x^2} + \frac{2x^2u}{1+x^2} \leq -\frac{2x^2}{1+x^2} + 2|u| = -(1 - e^{-V(x)}) + 2|u|.$$

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Theorem (E. Sontag, 1998)

Finite-dimensional bilinear 0-GAS systems are iISS.

$$\dot{x}(t) = Ax(t) + Bu(t) + C(x(t), u(t)). \quad (\text{BL})$$

Theorem (A. Mironchenko, H. Ito)

The following statements are equivalent (for bounded B and C)

- (BL) is iISS
- (BL) is 0-UGASS
- $W : x \mapsto \ln \left(1 + \int_0^\infty \|T(t)x\|^2 dt \right)$ is a (in general) non-coercive iISS Lyapunov function for (BL).

$$X \text{ is Hilbert space} \quad \Rightarrow \quad W(x) = \ln \left(1 + \langle Px, x \rangle \right),$$

$$\langle Ax, Px \rangle + \langle Px, Ax \rangle = -\|x\|_X^2, \quad \forall x \in D(A).$$

$$P \text{ is coercive} \quad \Rightarrow \quad W \text{ is coercive}$$

$$\begin{cases} \frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial l^2} + ax - x \left(\frac{\partial x}{\partial l} \right)^2 + u, & \forall t > 0, \\ x(0, t) = x(\pi, t) = 0. \end{cases}$$

- $X = H_0^1(0, \pi)$.

1 If $U = W_0^{1, \frac{4}{3}}(0, \pi) \Rightarrow$ ISS iff $a \leq 1$.

2 If $U = L_2(0, \pi) \Rightarrow$ ISS provided $a < 1$.

$$V(x) = \int_0^\pi \left(\frac{\partial x}{\partial l} \right)^2 dl = \|x\|_{H_0^1(0,\pi)}^2.$$

The derivative of V along the trajectories is equal

$$\begin{aligned} \dot{V}(x) = & -2 \int_0^\pi \left(\frac{\partial^2 x}{\partial l^2} \right)^2 dl + 2a \int_0^\pi \left(\frac{\partial x}{\partial l} \right)^2 dl \\ & - \frac{2}{3} \int_0^\pi \left(\frac{\partial x}{\partial l} \right)^4 dl - 2 \int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl. \end{aligned}$$

$$V(x) = \int_0^\pi \left(\frac{\partial x}{\partial l} \right)^2 dl = \|x\|_{H_0^1(0,\pi)}^2.$$

The derivative of V along the trajectories is equal

$$\begin{aligned} \dot{V}(x) = & -2 \int_0^\pi \left(\frac{\partial^2 x}{\partial l^2} \right)^2 dl + 2a \int_0^\pi \left(\frac{\partial x}{\partial l} \right)^2 dl \\ & - \frac{2}{3} \int_0^\pi \left(\frac{\partial x}{\partial l} \right)^4 dl - 2 \int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl. \end{aligned}$$

$$U = L_2(0, \pi) \Rightarrow -2 \int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl \leq 2\omega \int_0^\pi \left(\frac{\partial^2 x}{\partial l^2} \right)^2 dl + 2\frac{1}{\omega} \int_0^\pi u^2 dl$$

$$V(x) = \int_0^\pi \left(\frac{\partial x}{\partial l} \right)^2 dl = \|x\|_{H_0^1(0,\pi)}^2.$$

The derivative of V along the trajectories is equal

$$\begin{aligned} \dot{V}(x) = & -2 \int_0^\pi \left(\frac{\partial^2 x}{\partial l^2} \right)^2 dl + 2a \int_0^\pi \left(\frac{\partial x}{\partial l} \right)^2 dl \\ & - \frac{2}{3} \int_0^\pi \left(\frac{\partial x}{\partial l} \right)^4 dl - 2 \int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl. \end{aligned}$$

$$U = L_2(0, \pi) \Rightarrow -2 \int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl \leq 2\omega \int_0^\pi \left(\frac{\partial^2 x}{\partial l^2} \right)^2 dl + 2 \frac{1}{\omega} \int_0^\pi u^2 dl,$$

$$U = W_0^{1, \frac{4}{3}}(0, \pi) \Rightarrow -2 \int_0^\pi \frac{\partial^2 x}{\partial l^2} u dl \leq \frac{\omega}{4} \int_0^\pi \left| \frac{\partial x}{\partial l} \right|^4 dl + \frac{1}{\omega^{\frac{1}{3}}} \frac{3}{4} \int_0^\pi \left| \frac{\partial u}{\partial l} \right|^{\frac{4}{3}} dl$$

Case $U = L_2(0, \pi)$

$$\dot{V}(x) \leq -2(1 - a - \omega)V(x) - \frac{2}{3\pi}V^2(x) + 2\frac{1}{\omega} \int_0^\pi u^2 dl.$$

- $a < 1 \Rightarrow$ ISS for $U = L_2(0, \pi)$.

Case $U = L_2(0, \pi)$

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Case $U = W_0^{1, \frac{4}{3}}(0, \pi)$

$$\dot{V}(x) \leq -2(1 - a)V(x) - \left(\frac{1}{3} - \frac{\omega}{2}\right)\frac{1}{\pi}V^2(x) + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{2}\|u\|_{W_0^{1, \frac{4}{3}}(0, \pi)}^{\frac{4}{3}}.$$

- $a \leq 1 \Rightarrow$ ISS for $U = W_0^{1, \frac{4}{3}}(0, \pi)$

Case $U = L_2(0, \pi)$

$$\dot{V}(x) \leq -2(1 - a - \omega)V(x) - \frac{2}{3\pi}V^2(x) + 2\frac{1}{\omega} \int_0^\pi u^2 dl.$$

- $a < 1 \Rightarrow$ ISS for $U = L_2(0, \pi)$.

Case $U = W_0^{1, \frac{4}{3}}(0, \pi)$

$$\dot{V}(x) \leq -2(1 - a)V(x) - \left(\frac{1}{3} - \frac{\omega}{2}\right)\frac{1}{\pi}V^2(x) + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{2}\|u\|_{W_0^{1, \frac{4}{3}}(0, \pi)}^{\frac{4}{3}}.$$

- $a \leq 1 \Rightarrow$ ISS for $U = W_0^{1, \frac{4}{3}}(0, \pi)$

Σ is no more 0-UGASs for $a > 1$.

I do not know whether Σ is ISS for $U = L_2(0, \pi)$ for $a = 1$ or not.

ISS Lyapunov functions for Sobolev state spaces

$$\frac{\partial x}{\partial t} = c \frac{\partial^2 x}{\partial l^2} + f(x(l, t), \frac{\partial x}{\partial l}(l, t)) + u(l, t), \quad \forall t > 0$$

$$x(0, t) = x(L, t) = 0, \quad \forall t \geq 0.$$

Theorem (A.M., Hiroshi Ito, 2014, submitted to SICON)

Let $X = W_0^{1,2q}(0, L)$ and $\forall x \in X$, some convex $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$ and some $\epsilon > 0$

$$\int_0^L \left(\frac{\partial x}{\partial l} \right)^{2q-2} \frac{\partial^2 x}{\partial l^2} f(x, \frac{\partial x}{\partial l}) dl \geq \int_0^L \eta \left(\left(\frac{\partial x}{\partial l} \right)^{2q} \right) dl$$

$$\hat{\alpha}(s) := \frac{\pi^2}{q^2 L^2} (c - \epsilon) s + L \eta \left(\frac{s}{L} \right) \geq 0, \quad \forall s \in \mathbb{R}_+.$$

$$\Rightarrow V(x) = \int_0^L \left(\frac{\partial x}{\partial l} \right)^{2q} dl \quad \text{is ISS-LF} \quad \text{w.r.t} \quad U = L_{2q}(0, L)$$

$$\frac{\partial x}{\partial t} = c \frac{\partial^2 x}{\partial l^2} + f(x(l, t), u(l, t)), \quad \forall t > 0$$

$$x(0, t) \frac{\partial x}{\partial l}(0, t) = x(L, t) \frac{\partial x}{\partial l}(L, t) = 0, \quad \forall t \geq 0$$

Theorem (A.M., Hiroshi Ito, 2014, submitted to SICON)

Let $X = L_{2q}(0, L)$ and let $\exists \alpha \in \mathcal{K}_\infty \cup \{0\}$, $\sigma \in \mathcal{K}$: $W : y \mapsto y^{2q}$ satisfies

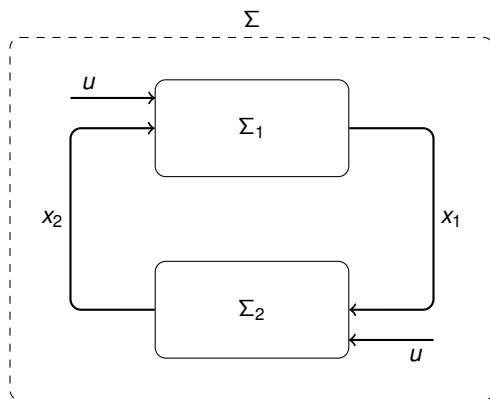
$$\dot{W}(y) \leq -\alpha(W(y)) + W(y)\sigma(|u|)$$

Let any of the following conditions hold:

- 1 $x(0, t) = 0$ for all $t \geq 0$ or $x(L, t) = 0$ for all $t \geq 0$.
- 2 α is convex and \mathcal{K}_∞ .

$\Rightarrow V(x) = \ln(1 + \|x\|_{L_{2q}(0, L)}^{2q})$ is iISS-LF w.r.t. $U = H_0^1(0, L)$.

Interconnections of 2 systems



$$\text{gain}_{\Sigma_1 \rightarrow \Sigma_2} \circ \text{gain}_{\Sigma_2 \rightarrow \Sigma_1}(s) < s \Rightarrow \text{iISS of } \Sigma$$

$$\Sigma : \begin{cases} \Sigma_1 : & \dot{x}_1 = A_1 x_1 + f_1(x_1, x_2, u), \quad x_1 \in X_1 \\ \Sigma_2 : & \dot{x}_2 = A_2 x_2 + f_2(x_1, x_2, u), \quad x_2 \in X_2 \end{cases}$$

ISS-LF for Σ_i

$V_i : X_i \rightarrow \mathbb{R}_+$ is **ISS-Lyapunov functions for Σ_i** , $i = 1, 2$ iff

- $V_1(x_1) \geq \max \{ \chi_{12}(V_2(x_2)), \chi_1(\|u(0)\|_U) \} \Rightarrow \dot{V}_1(x_1) \leq -\alpha_1(V_1(x_1)),$
- $V_2(x_2) \geq \max \{ \chi_{21}(V_1(x_1)), \chi_2(\|u(0)\|_U) \} \Rightarrow \dot{V}_2(x_2) \leq -\alpha_2(V_2(x_2)),$

Small-gain theorem for two interconnected systems

Theorem (Dashkovskiy, M., MCSS, 2013)

Let V_1, V_2 be ISS-Lyapunov function for Σ_1, Σ_2 with gains χ_{12}, χ_{21} .

$$\chi_{12} \circ \chi_{21} < id \quad (\text{SGC})$$

\Rightarrow

- Σ is ISS.
- $V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}$ is a Lyapunov function for Σ .

For 2 finite-dimensional systems:

Jiang, Mareels, Wang, Automatica, 1996.

Interconnections of n systems:

Dashkovskiy, Rüffer, Wirth, MTNS 2006, SICON 2010.

Example: interconnected ISS parabolic systems

$$\begin{cases} \frac{\partial x_1}{\partial t} = q_1 \frac{\partial^2 x_1}{\partial l^2} + x_2^2, \\ x_1(0, t) = x_1(\pi, t) = 0; \\ \frac{\partial x_2}{\partial t} = q_2 \frac{\partial^2 x_2}{\partial l^2} + \sqrt{|x_1|}, \\ x_2(0, t) = x_2(\pi, t) = 0. \end{cases}$$

State spaces:

$$X_1 := L_2(0, \pi) \quad X_2 := L_4(0, \pi)$$

ISS Lyapunov functions:

$$V_1(x_1) = \int_0^\pi x_1^2(l) dl = \|x_1\|_{L_2(0, \pi)}^2,$$

$$V_2(x_2) = \int_0^\pi x_2^4(l) dl = \|x_2\|_{L_4(0, \pi)}^4.$$

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$$V_2(x_2) = \int_0^\pi x_2^4(l) dl = \|x_2\|_{L_4(0,\pi)}^4.$$

Lie derivative of V_1 :

$$\begin{aligned} \frac{d}{dt} V_1(x_1) &= 2 \int_0^\pi x_1(l, t) \left(q_1 \frac{\partial^2 x_1}{\partial l^2}(l, t) + x_2^2(l, t) \right) dl \\ &\leq -2q_1 \left\| \frac{dx_1}{dl} \right\|_{L_2(0,\pi)}^2 + 2 \|x_1\|_{L_2(0,\pi)} \|x_2\|_{L_4(0,\pi)}^2. \end{aligned}$$

By Friedrichs' inequality

$$\begin{aligned} \frac{d}{dt} V_1(x_1) &\leq -2q_1 \|x_1\|_{L_2(0,\pi)}^2 + 2 \|x_1\|_{L_2(0,\pi)} \|x_2\|_{L_4(0,\pi)}^2 \\ &= -2q_1 V_1(x_1) + 2\sqrt{V_1(x_1)}\sqrt{V_2(x_2)}. \end{aligned}$$

Take

$$\chi_{12}(r) = \frac{1}{q_1^2(1-\varepsilon)^2} r, \quad \forall r > 0,$$

with arbitrary $a > 0$. We obtain

$$V_1(x_1) \geq \chi_{12}(V_2(x_2)) \quad \Rightarrow \quad \frac{d}{dt} V_1(x_1) \leq -2\varepsilon q_1 V_1(x_1).$$

Analogously for

$$\chi_{21}(r) = \left(\frac{3}{4} q_2(1-\varepsilon)\right)^{-4} r, \quad \forall r > 0,$$

we have

$$V_2(x_2) \geq \chi_{21}(V_1(x_1)) \quad \Rightarrow \quad \frac{d}{dt} V_2(x_2) \leq -3\varepsilon q_2 V_2(x_2).$$

$$\chi_{12} \circ \chi_{21} < \text{Id} \quad \Leftrightarrow \quad q_1^2 \left(\frac{3}{4} q_2\right)^4 > 1.$$

$$\Sigma : \begin{cases} \Sigma_1 : & \dot{x}_1 = A_1 x_1 + f_1(x_1, x_2, u), \quad x_1 \in X_1 \\ \Sigma_2 : & \dot{x}_2 = A_2 x_2 + f_2(x_1, x_2, u), \quad x_2 \in X_2 \end{cases}$$

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$V_i : X_i \rightarrow \mathbb{R}_+$ is **iISS-Lyapunov functions** for Σ_i , $i = 1, 2$ iff

- $\dot{V}_1(x_1) \leq -\alpha_1(\|x_1\|_{X_1}) + \sigma_1(\|x_2\|_{X_2}) + \kappa_1(\|u(0)\|_U)$,
- $\dot{V}_2(x_2) \leq -\alpha_2(\|x_2\|_{X_2}) + \sigma_2(\|x_1\|_{X_1}) + \kappa_2(\|u(0)\|_U)$,

Theorem (A. Mironchenko, H. Ito)

Let:

- $\lim_{s \rightarrow \infty} \alpha_i(s) = \infty$ or $\lim_{s \rightarrow \infty} \sigma_{3-i}(s) \kappa_i(1) < \infty$ for $i=1,2$.

- $\exists c > 1: \forall s \in \mathbb{R}_+: \psi_{11}^{-1} \circ \psi_{12} \circ \alpha_1^\ominus \circ c\sigma_1 \circ \psi_{21}^{-1} \circ \psi_{22} \circ \alpha_2^\ominus \circ c\sigma_2(s) \leq s$.

$\Rightarrow \Sigma$ is iISS.

If additionally

- $\alpha_i \in \mathcal{K}_\infty$ for $i = 1, 2 \Rightarrow \Sigma$ is ISS.

iISS (ISS) Lyapunov function:

$$V(x) = \int_0^{V_1(x_1)} \lambda_1(s) ds + \int_0^{V_2(x_2)} \lambda_2(s) ds.$$

$$\left\{ \begin{array}{l} \frac{\partial x_1}{\partial t}(l, t) = \frac{\partial^2 x_1}{\partial l^2}(l, t) + x_1(l, t)x_2^4(l, t), \\ x_1(0, t) = x_1(\pi, t) = 0; \\ \frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2 \left(\frac{\partial x_2}{\partial l} \right)^2 + \left(\frac{x_1^2}{1+x_1^2} \right)^{\frac{1}{2}}, \\ x_2(0, t) = x_2(\pi, t) = 0. \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial x_1}{\partial t}(l, t) = \frac{\partial^2 x_1}{\partial l^2}(l, t) + x_1(l, t)x_2^4(l, t), \\ x_1(0, t) = x_1(\pi, t) = 0; \\ \frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2 \left(\frac{\partial x_2}{\partial l} \right)^2 + \left(\frac{x_1^2}{1+x_1^2} \right)^{\frac{1}{2}}, \\ x_2(0, t) = x_2(\pi, t) = 0. \end{array} \right.$$

For what a, b is this system UGASs?

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For what a, b is this system UGASs?

State spaces: $X_1 := L_2(0, \pi)$ and $X_2 := H_0^1(0, \pi)$

Strategy

- 1 x_1 -subsystem is iISS
- 2 x_2 -subsystem is ISS
- 3 Interconnection is UGASs

$$\begin{cases} \frac{\partial x_1}{\partial t}(l, t) = \frac{\partial^2 x_1}{\partial l^2}(l, t) + x_1(l, t)x_2^4(l, t), \\ x_1(0, t) = x_1(\pi, t) = 0; \end{cases}$$

iISS-Lyapunov function:

$$V_1(x_1) := \ln \left(1 + \|x_1\|_{L_2(0,\pi)}^2 \right)$$

Lie derivative of V_1 :

$$\dot{V}_1 \leq -\frac{2\|x_1\|_{L_2(0,\pi)}^2}{1 + \|x_1\|_{L_2(0,\pi)}^2} + 2\|x_2\|_{L_\infty(0,\pi)}^4.$$

Finally, via **Agmon's inequality**:

$$\dot{V}_1(x_1) \leq -\frac{2\|x_1\|_{L_2(0,\pi)}^2}{1 + \|x_1\|_{L_2(0,\pi)}^2} + 8\|x_2\|_{H_0^1(0,\pi)}^4.$$

$$\begin{cases} \frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2 \left(\frac{\partial x_2}{\partial l} \right)^2 + \overbrace{\left(\frac{x_1^2}{1+x_1^2} \right)^{\frac{1}{2}}}^u, \\ x_2(0, t) = x_2(\pi, t) = 0. \end{cases}$$

ISS-Lyapunov function:

$$V_2(x_2) = \int_0^\pi \left(\frac{\partial x_2}{\partial l} \right)^2 dl = \|x_2\|_{H_0^1(0,\pi)}^2.$$

Lie derivative of V_2 :

$$\dot{V}_2(x_2) \leq -2(1 - a - \omega)V_2(x_2) - \frac{2b}{3\pi}V_2^2(x_2) + 2\frac{1}{\omega} \int_0^\pi u^2 dl.$$

$$\dot{V}_2 \leq -2\left(1 - a - \frac{\omega}{2}\right)\|x_2\|_{H_0^1(0,\pi)}^2 - \frac{2b}{3\pi}\|x_2\|_{H_0^1(0,\pi)}^4 + \frac{\pi}{\omega} \left(\frac{\|x_1\|_{L_2(0,\pi)}^2}{1 + \|x_1\|_{L_2(0,\pi)}^2} \right).$$

Interconnection is UGASs

Condition for UGASs: for some $c > 0$, for all $s \in \mathbb{R}_+$

$$\psi_{11}^{-1} \circ \psi_{12} \circ \alpha_1^\ominus \circ c\sigma_1 \circ \psi_{21}^{-1} \circ \psi_{22} \circ \alpha_2^\ominus \circ c\sigma_2(s) \leq s$$

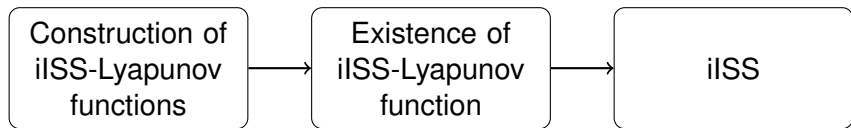
In our case: $\psi_{11}(s) = \psi_{12}(s) = \ln(1 + s^2)$, $\psi_{21}(s) = \psi_{22}(s) = s^2$

$$\alpha_1(s) = \frac{2s^2}{1 + s^2}, \quad \sigma_1(s) = 8s^4, \quad \kappa_1(s) = 0$$

$$\alpha_2(s) = 2 \left(1 - a - \frac{\omega}{2}\right) s^2 + \frac{2b}{3\pi} s^4, \quad \sigma_2(s) = \frac{\pi}{\omega} \left(\frac{s^2}{1 + s^2}\right), \quad \kappa_2(s) = 0$$

Condition for UGASs

$$a + \frac{3\pi^2}{b} < 1, \quad b \geq 0.$$



Discussed results

- Bilinear systems
- ISS of parabolic systems over Sobolev spaces
- iISS of parabolic systems over L_q spaces
- ISS/iISS small-gain theorems
- Interconnections of 1-dim parabolic systems

Possible research directions

Linear + Bilinear systems

- Difference between iISS and ISS for linear systems with unbounded B .
- Lyapunov theory for $\dot{x}(t) = Ax(t) + Bu(t)$, with unbounded B .
- Bilinear systems with unbounded bilinear operator.

Characterizations of ISS

- Lyapunov characterization
- Characterization via various robust stability notions

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Thank you for attention!