Lyapunov functions for input-to-state stability of infinite-dimensional systems with integrable inputs

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Class of systems

Definition

The triple $\Sigma = (X, \mathcal{U}, \phi)$, $\quad \phi : \mathbb{R}_+ \times X \times \mathcal{U} \to X$ is called control system, if:

(Σ1) Forward-completeness: for every $x \in X$, $u \in \mathcal{U}$ and for all $t \geq 0$ the value $\phi(t, x, u) \in X$ is well-defined.

(Σ2) Continuity: for each $(x, u) \in X \times \mathcal{U}$ the map $t \mapsto \phi(t, x, u)$ is continuous.

(Σ3) Cocycle property: for all $t, h \geq 0$, for all $x \in X$, $u \in \mathcal{U}$ we have

$$\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u).$$

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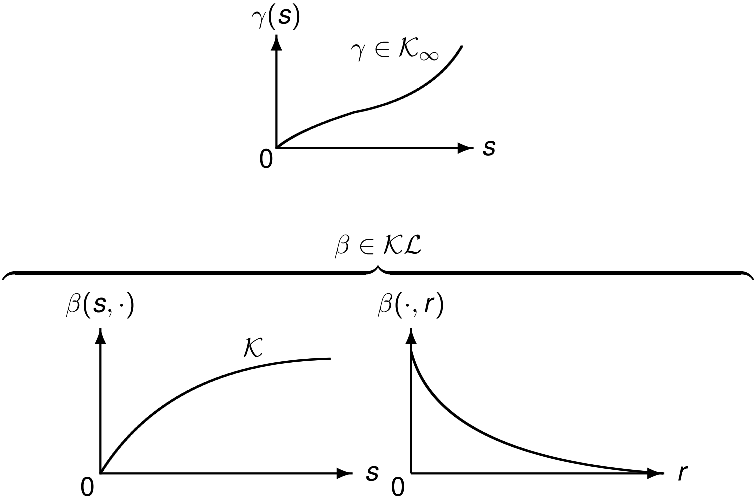
Examples

- Ordinary differential equations
- Evolution Partial differential equations with Lipschitz nonlinearities
- Broad classes of boundary control systems
- Time-delay systems
- Heterogeneous systems with distinct components

Comparison functions



Input-to-state stability

Definition (Sontag, 1989, for ODEs)

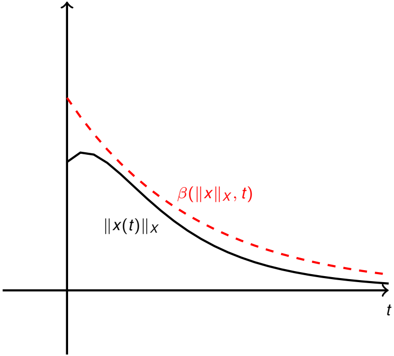
$$\text{ISS} \quad :\Leftrightarrow \quad \|x(t)\|_X \le \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

increasing in $\|x\|_X$

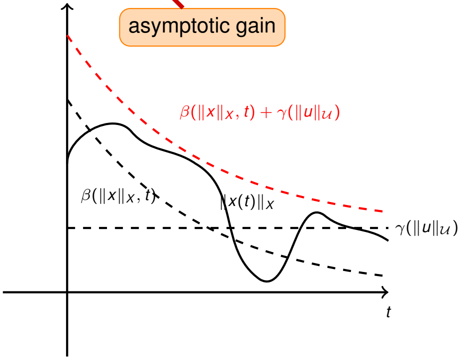
decreasing to 0 in t

$\gamma(0) = 0$, increasing

asymptotic gain



(a) $u \equiv 0$



(b) $u \not\equiv 0$

Motivation

Why ISS?

1. Unified theory of internal and external stability
2. Robust control & observation of nonlinear systems
3. Analysis & control of nonlinear networks
4. …

- For ODEs and delay systems the usual choice is $\mathcal{U} := L_\infty(\mathbb{R}_+, U)$.
- For PDEs $\mathcal{U} := L_p(\mathbb{R}_+, U)$ is of interest as well.

For a survey for a broad literature on ISS we refer to

- Sontag. *Input to state stability: basic concepts and results.* In Nonlinear and Optimal Control Theory, chapter 3, 2008.
- M., Prieur. *Input-to-state stability of infinite-dimensional systems: recent results and open questions.* To appear in SIAM Review, 2020.

Definition (ISS Lyapunov functions)

$V \in C(X, \mathbb{R}_+)$ is a non-coercive ISS Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \neq 0$

(ii) $\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u\|_\mathcal{U}) \quad \forall x \in X, \forall u \in \mathcal{U}$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}\big(V(\phi(t, x, u)) - V(x)\big).$$

V is called a coercive ISS Lyapunov function if

$$\exists \psi_1, \psi_2 \in \mathcal{K}_\infty : \quad \psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \neq 0.$$

Direct coercive and non-coercive ISS Lyapunov theorems

- Dashkovskiy, M. *Input-to-state stability of infinite-dimensional control systems*, MCSS, 2013.
- Jacob, M., Partington, Wirth. *Non-coercive Lyapunov functions for input-to-state stability of infinite-dimensional systems.* Submitted to SICON, 2019.
- Zheng, Zhu. *Input-to-state stability with respect to boundary disturbances for a class of semi-linear parabolic equations.* Automatica, 2018.
- Prieur, Mazenc. *ISS Lyapunov functions for time-varying hyperbolic systems of balance laws.* MCSS, 2012.

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Why non-coercive ISS Lyapunov functions?

- Classical constructions of Lyapunov functions often lead to non-coercive ones
- There are PDEs with boundary controls (as a heat equation with Dirichlet inputs), for which only non-coercive ISS Lyapunov functions are known.

Equivalent definition of the ISS Lyapunov function

Definition

$V : X \to \mathbb{R}_+$ is a non-coercive ISS Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

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- The definition is very general and formally applies to any input space $\mathcal{U}$
- If $\mathcal{U} = L_\infty(\mathbb{R}_+, U)$, $PC_b(\mathbb{R}_+, U)$, etc., then this definition is very natural.

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What about $\mathcal{U} = L_p(\mathbb{R}_+, U)$?

Equivalent definition of an ISS Lyapunov function

For all $u \in \mathcal{U}$, $\tau \geq 0$ define $$u_\tau(s) := \begin{cases} u(s) & , \text{ if } s \in [0,\tau], \\ 0 & , \text{ if } s > \tau. \end{cases}$$

Lemma (Equivalent definition of an ISS Lyapunov function)

Let $\mathcal{U} = L_p(\mathbb{R}_+, U)$, $p \in [1, +\infty]$.

$V \in C(X, \mathbb{R}_+)$ *is a non-coercive ISS Lyapunov function* $\Leftrightarrow \exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \neq 0.$

(ii) $\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\inf_{\tau>0} \|u_\tau\|_{\mathcal{U}}) \quad \forall x \in X, \forall u \in \mathcal{U}.$

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What about $\mathcal{U} = L_p(\mathbb{R}_+, U)$?

As $\inf_{\tau \geq 0} \|u_\tau\|_{\mathcal{U}} = 0$ for all $u \in L_p(\mathbb{R}_+, U)$, by previous lemma V satisfies

$$\dot{V}_u(x) \leq -\alpha(\|x\|_X), \quad u \in L_p(\mathbb{R}_+, U),$$

which is a too strong property.

Equivalent definition of an ISS Lyapunov function

For all $u \in \mathcal{U}$, $\tau \geq 0$ define $\quad u_\tau(s) := \begin{cases} u(s) & , \text{ if } s \in [0,\tau], \\ 0 & , \text{ if } s > \tau. \end{cases}$

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(i) $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \neq 0.$

(ii) $\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\inf_{\tau>0} \|u_\tau\|_{\mathcal{U}}) \quad \forall x \in X, \forall u \in \mathcal{U}.$

Other definition of an ISS Lyapunov function for L_p-spaces is needed.

L_p-ISS Lyapunov function

Definition (L_p-ISS Lyapunov function)

$V \in C(X, \mathbb{R}_+)$ is a non-coercive L_p-ISS Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \neq 0,$

(ii) $\dot{V}_u(x) \leq -\alpha(\|x\|_X) + c\|u(0)\|_U^p \quad \forall x \in X, \forall u \in C(\mathbb{R}_+, U),$

V is called a coercive L_p-ISS Lyapunov function if

$$\exists \psi_1, \psi_2 \in \mathcal{K}_\infty : \quad \psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \neq 0.$$

Direct coercive L_p-ISS Lyapunov theorem

Theorem (Direct coercive L_p-ISS Lyapunov theorem (M., IFAC WC 2020))

Let:

- $\mathcal{U} := L_{p,loc}(\mathbb{R}_+, U)$, *where* $p \in [1, +\infty)$ *and* U *is a normed linear space.*
- $\Sigma := (X, \mathcal{U}, \phi)$ *be forward complete.*
- ϕ *is continuous w.r.t. inputs (in* $\mathcal{U}$*-norm).*

Let V be an L_p-ISS Lyapunov function for Σ with $\alpha \in \mathcal{K}_\infty$.
Then:

(i) Σ *is* L_q*-ISS for all* $q \in [p, +\infty)$.

(ii) *For each* $z \geq 0$ *the map* $V_z : X \to \mathbb{R}_+$, *defined by*

$$V_z(x) = \int_0^{V(x)} \big(\alpha(s)\big)^z ds, \quad x \in X,$$

is an $L_{(z+1)p}$*-ISS Lyapunov function for* Σ.

Non-coercive ISS Lyapunov theorems

Definition

Let $\Sigma = (X, \mathcal{U}, \phi)$ be given.

- We call $0 \in X$ an equilibrium point (of the undisturbed system) if $\phi(t, 0, 0) = 0$ for all $t \geq 0$.
- We say Σ has the CEP property, if 0 is an equilibrium and for every $\varepsilon > 0$ and for any $h > 0$ there exists a $\delta = \delta(\varepsilon, h) > 0$, so that

$$t \in [0, h], \; \|x\|_X \leq \delta, \; \|u\|_{\mathcal{U}} \leq \delta \;\Rightarrow\; \|\phi(t, x, u)\|_X \leq \varepsilon. \tag{1}$$

- Σ has bounded reachability sets (BRS), if:

$$C > 0, \; \tau > 0 \quad \Rightarrow \quad \sup_{\|x\|_X \leq C, \; \|u\|_{\mathcal{U}} \leq C, \; t \in [0, \tau]} \|\phi(t, x, u)\|_X < \infty.$$

Direct non-coercive Lyapunov theorem

Theorem (Direct non-coercive L_p-ISS Lyapunov theorem (M., IFAC WC 2020))

Let:

- $\mathcal{U} := L_{p,loc}(\mathbb{R}_+, U)$, *where* $p \in [1, +\infty)$ *and* U *is a normed linear space.*
- $\Sigma := (X, \mathcal{U}, \phi)$ *be forward complete.*
- *for some* $q \geq p$ *the system* Σ *is* L_q*-BRS and* L_q*-CEP.*
- ϕ *is continuous w.r.t. inputs (in* $\mathcal{U}$*-norm).*

$$\exists \text{ a non-coercive } L_p\text{-ISS Lyapunov function for } \Sigma \quad \Rightarrow \quad \Sigma \text{ is } L_r\text{-ISS for all } r \geq q.$$

Proof relies deeply on characterizations of ISS for infinite-dimensional systems, obtained in

- M. *Local input-to-state stability: Characterizations and counterexamples.* Sys. & Con. Lett., 2016.
- M., Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems.* IEEE TAC, 2018.

Benchmark: Heat equation with Dirichlet boundary input

For $\mu > 0$, consider the following equation

$$x_t(t,z) = \mu x_{zz}(t,z), \tag{2}$$

subject to Dirichlet boundary conditions:

$$x(t,0) = u(t), \quad x(t,1) = 0, \quad t > 0. \tag{3}$$

This system is ISS for $X := L_2(0,1)$ and $\mathcal{U} := L_\infty(\mathbb{R}_+, \mathbb{R})$, as shown via:

✓ Admissibility approach

- Jacob, Nabiullin, Partington, Schwenninger. *Infinite-dimensional input-to-state stability and Orlicz spaces.* SICON, 2018.

✓ Spectral analysis

- Karafyllis, Krstic. *ISS with respect to boundary disturbances for 1-D parabolic PDEs.* IEEE TAC, 2016.

✓ Monotonicity methods

- M., Karafyllis, Krstic. *Monotonicity methods for input-to-state stability of nonlinear parabolic PDEs with boundary disturbances.* SICON, 2019.

? No coercive ISS Lyapunov functions are known

Benchmark: Heat equation with Dirichlet boundary input

For $\mu > 0$, consider the following equation

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subject to Dirichlet boundary conditions:

$$x(t, 0) = u(t), \quad x(t, 1) = 0, \quad t > 0. \tag{3}$$

Assume that $X := L_2(0, 1)$ and $\mathcal{U} := L_2(\mathbb{R}_+, \mathbb{R})$.

We are going to show that $V : X \to \mathbb{R}_+$ defined by

$$V(x) := \int_0^1 zx^2(z)dz$$

is a non-coercive L_2-ISS Lyapunov function for (2)-(3).

- V is non-coercive:

$$0 < V(x) = \int_0^1 z x^2(z) dz \leq \|x\|^2_{L_2(0,1)}, \quad \text{for all } x \neq 0.$$

- V is continuous on X.
- Dissipation inequality (for smooth enough u):

$$\dot{V}_u(x) \quad \leq \quad -2\mu \int_0^1 z x_z^2(z) dz - 2\mu \int_0^1 x(z) x_z(z) dz.$$

- V is non-coercive:

$$0 < V(x) = \int_0^1 zx^2(z)dz \le \|x\|^2_{L_2(0,1)}, \quad \text{for all } x \neq 0.$$

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Using the weighted Poincaré inequality:

$$\begin{aligned} \dot{V}_u(x) &\le -4\mu \int_0^1 zx^2(z)dz - \mu x^2(z)\big|_{z=0}^{z=1} \\ &= -4\mu V(x) + \mu u^2(0). \end{aligned}$$

- V is non-coercive:

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V is a non-coercive L_2-ISS Lyapunov function.

Benchmark: Heat equation with Dirichlet boundary input

For $\mu > 0$, consider the following equation

$$x_t(t,z) = \mu x_{zz}(t,z), \tag{4}$$

subject to Dirichlet boundary conditions:

$$x(t,0) = u(t), \quad x(t,1) = 0, \quad t > 0. \tag{5}$$

We have shown

- $V(x) := \int_0^1 z x^2(z) dz$ is a non-coercive L_2-ISS Lyapunov function for (4)-(5).

At the same time:

- No coercive ISS Lyapunov functions are known.
- However, (4)-(5) is not L_2-ISS, as follows from admissibility analysis of (4)-(5).
- L_q-BRS and L_q-CEP properties are valid only for a certain $q \in (2, +\infty)$.
- Only for $s \geq q$ we get L_s-ISS.
- Still, existence of a non-coercive L_2-ISS Lyapunov function indicates, that some stability properties are retained also for L_2-norm, provided the input is smooth enough.

We discussed

- Coercive L_p-ISS Lyapunov theorems
- Non-coercive L_p-ISS Lyapunov theorems
- Comparison with the definition of L_∞-ISS Lyapunov functions
- Non-coercive ISS Lyapunov functions for heat equation with Dirichlet boundary inputs.

References

- M., *Lyapunov functions for input-to-state stability of infinite-dimensional systems with integrable inputs.* IFAC WC, 2020.
- Jacob, M., Partington, Wirth. *Non-coercive Lyapunov functions for input-to-state stability of infinite-dimensional systems.* Submitted to SICON, 2019.

Conclusion

We discussed

- Coercive L_p-ISS Lyapunov theorems
- Non-coercive L_p-ISS Lyapunov theorems
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Thank you for Your attention!

Papers and slides can be found at
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