Well-posedness and robust stability of evolution equations

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A general class of systems (used at least since 1960s)

Definition

The triple $\Sigma = (X, \mathcal{U}, \phi)$, $\phi : D_\phi \subseteq \mathbb{R}_+ \times X \times \mathcal{U} \to X$ is called control system, if:

(Σ1) Local existence: for all $(x, u) \in X \times \mathcal{U}$ there is $t_m = t_m(x, u) \in (0, +\infty]$ such that

$$D_\phi \cap \big(\mathbb{R}_+ \times \{(x, u)\}\big) = [0, t_m) \times \{(x, u)\} \subset D_\phi.$$

(Σ2) Continuity: for each $(x, u) \in X \times \mathcal{U}$ the map $t \mapsto \phi(t, x, u)$ is continuous.

(Σ3) Cocycle property: for all $x \in X$, $u \in \mathcal{U}$, for all $t, h \geq 0$ so that $[0, t + h] \times \{(x, u)\} \subset D_\phi$

$$\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u).$$

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$$\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u).$$

Examples

- Ordinary differential equations
- Evolution PDEs with distributed inputs
- Broad classes of linear boundary control systems
- Time-delay systems
- Heterogeneous systems

Input-to-state stability

Definition (Sontag, 1989, for ODEs)

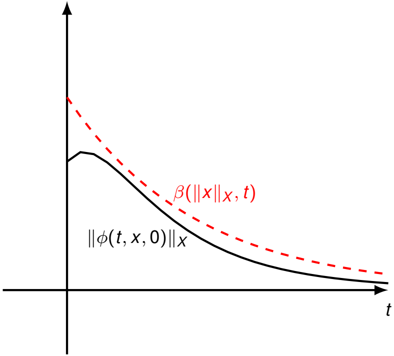
$$\text{ISS} \quad :\Leftrightarrow \quad \|\phi(t,x,u)\|_X \le \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall x, t, u.$$

increasing in $\|x\|_X$

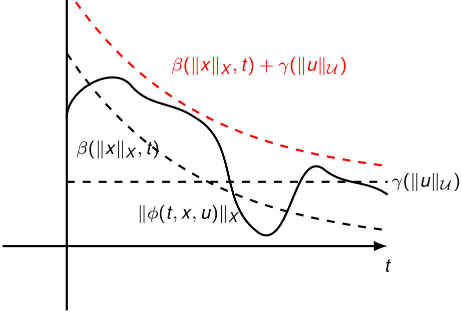
decreasing to 0 in t

$\gamma(0) = 0$, increasing

asymptotic gain



(a) $u \equiv 0$



(b) $u \not\equiv 0$

Toolbox for ISS analysis

- Direct and (partially) converse coercive Lyapunov theorems
  M., Wirth. *Lyapunov characterization of input-to-state stability for semilinear control systems over Banach spaces.* SCL, 2018.
- ISS superposition theorems
  M., Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems.* IEEE TAC, 2018.
- Noncoercive Lyapunov functions
  Jacob, M., Partington, Wirth. *Non-coercive Lyapunov functions for input-to-state stability of infinite-dimensional systems.* SICON, 2020.
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How to apply these tools for interesting PDE models?

How to analyze ISS

Tools for ISS analysis of nonlinear control systems

- Direct and converse coercive Lyapunov theorems
- ISS superposition theorems
- Noncoercive Lyapunov functions
- “Integral” characterizations
- Small-gain theorems

To make this toolbox working, we need to verify

- Well-posedness

How to analyze ISS

Tools for ISS analysis of nonlinear control systems

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To make this toolbox working, we need to verify

- Well-posedness
- Further desirable properties:
  - Boundedness-implies continuation
  - Forward completeness & Boundedness of reachability sets
  - Continuous dependence on initial states and inputs
  - Lipschitz regularity of the flow map

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We want to develop conditions guaranteeing these properties for semilinear infinite-dimensional systems.

Semilinear Evolution Equations (SEE)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_2 f(x(t), u(t)) + Bu(t), \quad t > 0 \\ x(0) &= x_0, \end{aligned}$$

Here:

- X is a Banach space
- U is a Banach space
- $\mathcal{U} := L^q(\mathbb{R}_+, U)$ (for most of the talk $\mathcal{U} := L^\infty(\mathbb{R}_+, U)$).
- $A : D(A) \subset X \to X$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X
- $f : X \times U \to X$
- $B : U \to X$ is a possibly unbounded operator, that belongs however to $L(U, X_{-1})$.
- $B_2 : X \to X$ is a possibly unbounded operator, that belongs however to $L(X, X_{-1})$.
- the extrapolation space X_{-1} is the closure of X in the norm $x \mapsto \|(aI - A)^{-1}x\|_X$

Semilinear Evolution Equations (SEE)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_2 f(x(t), u(t)) + Bu(t), \quad t > 0 \\ x(0) &= x_0, \end{aligned}$$

This (or closely related) class of systems was investigated in

- Natarajan, Bentsman. *Approximate local output regulation for nonlinear distributed parameter systems.* MCSS, 2016.
- Natarajan, Zhou, Weiss, Fridman. *Exact controllability of a class of nonlinear distributed parameter systems using back-and-forth iterations.* IJC, 2019.
- Schwenninger. *Input-to-state stability for parabolic boundary control: linear and semilinear systems*, 2020.
- Jacob, Dragan, Pritchard. *Infinite dimensional time varying systems with nonlinear output feedback.* IEOT, 1995.

Semilinear Evolution Equations (SEE)

$$\begin{aligned}\dot{x}(t) &= Ax(t) + f(x(t), u(t)) + Bu(t), \quad t > 0\\ x(0) &= x_0,\end{aligned}$$

Our objectives

- Derive sufficient conditions for
  - Local existence and uniqueness
  - Boundedness-implies continuation property
  - Forward completeness
  - Boundedness of reachability sets
  - Continuous dependence on initial states and inputs
  - Lipschitz regularity of the flow map
- Relationship to boundary control systems
- How this helps for the stability analysis?

Treatment of a linear part

Let us look closer at the linear systems with unbounded input operators

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in X, t > 0. \tag{1}$$

- A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X.
- $B \in L(U, X_{-1})$ for some Banach space U.
- X_{-1} is the completion of X w.r.t. $\|x\|_{X_{-1}} := \|(\beta - A)^{-1}x\|_X$, for some $\beta \in \rho(A)$.
- T extends uniquely to T_{-1} on X_{-1} whose generator A_{-1} is an extension of A.
- (1) is well-posed on X_{-1}: $\forall x_0 \in X$ and $\forall u \in L^1_{\mathrm{loc}}([0,\infty), U)$, the function $\phi(\cdot, x_0, u) : [0,\infty) \to X_{-1}$,

$$\phi(t, x_0, u) := T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds, \quad t \geq 0,$$

is called mild solution of (1).

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is called mild solution of (1).

That's great, but the trajectory is now in X_{-1}

$$\Sigma : \dot{x} = Ax + Bu.$$

(Global) well-posedness of linear systems for L^q-inputs

- $B \in L(U, X_{-1})$ is called a q-admissible control operator for $(T(t))_{t\geq 0}$, where $1 \leq q \leq \infty$, if

$$t \geq 0, \quad x_0 \in X, \quad u \in L^q([0,\infty), U) \quad \Rightarrow \quad \phi(t, x_0, u) \in X.$$

Proposition (Admissibility implies boundedness of reachability sets, Weiss, SICON 1989)

Let X, U be Banach spaces and let $p \in [1, \infty]$ be given. Then $B \in L(U, X_{-1})$ is q-admissible $\Leftrightarrow$ $\forall t > 0\ \exists h_t > 0$: for all $u \in L^q_{loc}(\mathbb{R}_+, U)$ we have $\Phi(t)u \in X$ and

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\, ds \right\|_X \leq h_t \|u\|_{L^q([0,t],U)}.$$

Semilinear Evolution Equations: assumptions on f and B

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x(t), u(t)) + Bu(t), \quad t > 0, \\ x(0) &= x_0, \end{aligned}$$

Assumption (forward completeness of a linear component)

The operator $B \in L(U, X_{-1})$ is ∞-admissible, and the map

$$(t, u) \mapsto \int_0^t T_{-1}(t-s)Bu(s)ds$$

is continuous on $\mathbb{R}_+ \times L^\infty(\mathbb{R}_+, U)$.

Assumption (Integrability of a nonlinearity)

For all $u \in L^\infty(\mathbb{R}_+, U)$ and any $x \in C(\mathbb{R}_+, X)$ the map $s \mapsto f\big(x(s), u(s)\big)$ is in $L^1_{loc}(\mathbb{R}_+, X)$.

Semilinear Evolution Equations: assumptions on f and B

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x(t), u(t)) + Bu(t), \quad t > 0, \\ x(0) &= x_0, \end{aligned}$$

Definition (Mild solutions)

A function $x \in C([0,\tau], X)$ is called a mild solution of (SEE) on $[0,\tau]$ corresponding to certain $x_0 \in X$ and $u \in L^\infty_{\text{loc}}(\mathbb{R}_+, U)$, if x solves the integral equation

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds + \int_0^t T(t-s)f\big(x(s), u(s)\big)ds.$$

Here the integrals are Bochner integrals of X-valued maps.

Assumption

The nonlinearity f satisfies the following properties:

(i) $f : X \times U \to X$ is Lipschitz continuous on bounded subsets of X:
for any $C > 0$ there is $L(C) > 0$, such that $\forall x, y \in B_C$, $\forall v \in B_{C,U}$ it holds that

$$\|f(y, v) - f(x, v)\|_X \leq L(C)\|y - x\|_X.$$

(ii) $f(x, \cdot)$ is continuous for all $x \in X$.

(iii) There exist $\sigma \in \mathcal{K}_\infty$ and $c > 0$ so that for all $u \in \mathcal{U}$ the following holds:

$$\|f(0, u)\|_X \leq \sigma(\|u\|_U) + c.$$

Define the r-ball around $Z \subset X$ by

$$B_r(Z) := \{y \in X : \mathrm{dist}\,(y, Z) < r\}.$$

For a set $\mathcal{S} \subset U$, denote the set of inputs with essential values in $\mathcal{S}$ as $\mathcal{U}_\mathcal{S}$.

$$\mathcal{U}_\mathcal{S} := \{u \in \mathcal{U} : u(t) \in \mathcal{S},\ \text{for a.e. } t \in \mathbb{R}_+\}.$$

Local existence and uniqueness

- Define $h_0 := \lim_{t\to+0} h_t$, where

$$\Big\|\int_0^t T_{-1}(t-s)Bu(s)\,ds\Big\|_X \leq h_t\|u\|_{L^\infty([0,t],U)}.$$

- Let T satisfy $\|T(t)\| \leq Me^{\lambda t}$ for all $t \geq 0$ and certain $M \geq 1$, $\lambda > 0$

Theorem (Picard-Lindelöf theorem: General case (AM, work in progress))

Pick any

- *bounded ball* $W = B_r(w) \subset X$
- *bounded set* $\mathcal{S} \subset U$
- $\delta > 0$

Then there is a time $t_1 = t_1(W, \mathcal{S}, \delta) > 0$, *such that*

- *For all* $x_0 \in W$ *and all* $u \in \mathcal{U}_S$ *there is a unique solution of (SEE) on* $[0, t_1]$
- *This solution lies in* $B_{Mr+h_0\|u\|_{L^\infty([0,t_1],U)}+\delta}(w)$.

Local existence and uniqueness

Corollary (Picard-Lindelöf theorem for zero-class admissible operators and quasi-contractive semigroups (AM, 2022))

Let B be zero-class admissible ($h_0 = 0$), and there is $\lambda > 0$ such that

$$\|T(t)\| \leq e^{\lambda t}, \quad t \geq 0.$$

Pick any

- *bounded ball $W = B_r(w) \subset X$*
- *bounded set $\mathcal{S} \subset U$*
- $\delta > 0$

Then there is a time $t_1 = t_1(W, \mathcal{S}, \delta) > 0$, such that

- *For all $x_0 \in W$ and all $u \in \mathcal{U}_{\mathcal{S}}$ there is a unique solution of (SEE) on $[0, t_1]$*
- *This solution lies in $B_{r+\delta}(w)$.*

If B is not zero-class ∞-admissible, or if T is not quasi-contractive, this strengthening does not hold in general.

Take-out-message I: Well-posedness

Theorem (Well-posedness and basic properties of SEE (AM, 2022))

Let

- X, U *be Banach spaces*
- $\mathcal{U} := L_\infty(\mathbb{R}_+, U)$ *be an input space*
- $s \mapsto f(x(s), u(s))$ *is integrable for continuous* $x(\cdot)$ *and* $u \in \mathcal{U}$.
- B *is* ∞*-admissible and* f *is Lipschitz continuous.*

Then

- *For each* $x \in X$ *and* $u \in \mathcal{U}$ *there is a unique maximal solution of (SEE) that we denote* $\phi(\cdot, x, u)$, *defined on* $[0, t_m(x, u))$.
- *The triple* $(X, \mathcal{U}, \phi)$ *is a control system. In particular,*

$$\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u).$$

- *If* $t_m(x, u)$ *is finite* $\quad \Rightarrow \quad \|\phi(t, x, u)\|_X \to \infty$ *as* $t \to t_m(x, u) - 0$.
- *If* $f(0, 0) = 0 \quad \Rightarrow \quad \phi$ *is continuous at the equilibrium in* (x, u).

Global existence

Definition

$(X, \mathcal{U}, \phi)$ is forward complete (FC), if for every $(x, u) \in X \times \mathcal{U}$ the trajectory $\phi(\cdot, x, u)$ is defined on $\mathbb{R}_+$.

Definition

$\Sigma = (X, \mathcal{U}, \phi)$ has bounded reachability sets (BRS), if for any $C > 0$ and any $\tau > 0$:

$$\sup \left\{ \|\phi(t, x, u)\|_X : \|x\|_X \leq C,\ \|u\|_{\mathcal{U}} \leq C,\ t \in [0, \tau] \right\} < \infty.$$

For nonlinear systems: FC $\not\Rightarrow$ BRS. For counterexample see

- M., Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems.* IEEE TAC, 2018.

Global existence

Theorem (Slightly restyled version of (Schwenninger, 2020))

Let

- X, U *be Banach spaces*
- $\mathcal{U} := L_\infty(\mathbb{R}_+, U)$ *be an input space*
- $s \mapsto f(x(s), u(s))$ *is integrable for continuous* $x(\cdot)$ *and* $u \in \mathcal{U}$.
- B *is* ∞*-admissible and* f *is globally Lipschitz continuous.*

Then

- $(X, \mathcal{U}, \phi)$ *is a forward complete control system satisfying BRS property.*

- Certainly, global Lipschitz continuity is a too strong requirement for the BRS property.
- A characterization of BRS property is strongly desirable.
  - Angeli, Sontag. *Forward completeness, unboundedness observability, and their Lyapunov characterizations*. SCL, 1999.
  - Karafyllis, Pepe, Chaillet, Wang. *Is global asymptotic stability necessarily uniform for time-delay systems?* Submitted, 2022.

Regularity of the flow map

Given BRS property, one can perform a regularity analysis of (SEE).

Definition

The flow of $(X, \mathcal{U}, \phi)$ is called Lipschitz continuous on compact intervals (for uniformly bounded inputs), if for any $\tau > 0$ and any $C > 0$ there exists $L > 0$ so that for any $x_1, x_2 \in \overline{B_C}$, for all $u \in B_{C,\mathcal{U}}$, it holds that

$$\|\phi(t, x_1, u) - \phi(t, x_2, u)\|_X \leq L\|x_1 - x_2\|_X, \quad t \in [0, \tau].$$

Theorem (Well-posedness and basic properties of SEE (AM, 2022))

Let

- *All assumptions we made for well-posedness hold.*
- *(SEE) is a control system satisfying BRS property.*

Then

- *(SEE) has a flow which is Lipschitz continuous on compact intervals for uniformly bounded inputs.*

Take-out-message II: Input-to-state stability

Theorem (Input-to-state stability of SEE (AM, 2022))

Let

- X, U *be Banach spaces*
- $\mathcal{U} := L_\infty(\mathbb{R}_+, U)$ *be an input space*
- $s \mapsto f(x(s), u(s))$ *is integrable for continuous* $x(\cdot)$ *and* $u \in \mathcal{U}$.
- B *is* ∞*-admissible and* f *is Lipschitz continuous.*

Then

- $\exists$ *coercive ISS-Lyapunov function* $\Rightarrow$ *ISS*
- $\exists$ *non-coercive ISS-Lyapunov function* $\wedge$ *BRS* $\Rightarrow$ *ISS*
- *ISS* $\Leftrightarrow$ *UAG* $\wedge$ *BRS* $\Leftrightarrow$ *ULIM* $\wedge$ *ULS* $\wedge$ *BRS*
- *ISS* $\Leftrightarrow$ *norm-to-integral ISS* $\wedge$ *BRS*

Proof.

Follows from above well-posedness results and infinite-dimensional ISS theory for general control systems. □

Take-out-message II: Input-to-state stability

Theorem (Input-to-state stability of SEE)

Let well-posedness assumptions hold.
Then

- $\exists$ *coercive ISS-Lyapunov function* $\Rightarrow$ *ISS*
- $\exists$ *non-coercive ISS-Lyapunov function* $\wedge$ *BRS* $\Rightarrow$ *ISS*
- *ISS* $\Leftrightarrow$ *UAG* $\wedge$ *BRS* $\Leftrightarrow$ *ULIM* $\wedge$ *ULS* $\wedge$ *BRS*
- *ISS* $\Leftrightarrow$ *norm-to-integral ISS* $\wedge$ *BRS*

References

- M., Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems.* IEEE TAC, 2018.
- Jacob, M., Partington, Wirth. *Non-coercive Lyapunov functions for input-to-state stability of infinite-dimensional systems.* SICON, 2020.
- M., Prieur. *Input-to-state stability of infinite-dimensional systems: recent results and open questions.* SIAM Review, 2020.

Why boundary control systems?

- We have developed a theory for evolution equations in Banach spaces
- However, a typical PDE control system looks like

$$x_t(z,t) = x_{zz}(z,t), \quad z \in (0,1), \ t > 0,$$
$$x(0,t) = 0, \quad x(1,t) = u(t), \quad t > 0.$$

- How to recast PDEs with boundary conditions into such abstract formulation?
- Theory of abstract boundary control systems provides a general way for this.

Linear boundary control systems

Let X and U be Banach spaces. Consider a system

$$\dot{x}(t) = \hat{A}x(t), \qquad x(0) = x_0,$$
$$\hat{R}x(t) = u(t),$$

- formal system operator $\hat{A} : D(\hat{A}) \subset X \to X$ is a linear operator,
- the control function u takes values in U,
- boundary operator $\hat{R} : D(\hat{R}) \subset X \to U$ is linear and satisfies $D(\hat{A}) \subset D(\hat{R})$.

Definition

This system is called a linear boundary control system (linear BCS) if:

(i) The operator $A : D(A) \to X$ with $D(A) = D(\hat{A}) \cap \ker(\hat{R})$ defined by

$$Ax = \hat{A}x \qquad \text{for} \quad x \in D(A)$$

is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X;

(ii) $\exists R \in \mathcal{L}(U, X)$: for all $u \in U$ we have $Ru \in D(\hat{A})$, $\hat{A}R \in \mathcal{L}(U, X)$ and

$$\hat{R}Ru = u, \qquad u \in U.$$

Semilinear boundary control systems

Definition (Semilinear boundary control system)

Let $(\hat{A}, \hat{R})$ be a linear BCS. Consider the following system

$$\begin{aligned}\dot{x}(t) &= \hat{A}x(t) + f(x(t), w(t)),\ \ t > 0,\\ \hat{R}x(t) &= u(t),\ \ t > 0,\\ x(0) &= x_0,\end{aligned}$$

with a nonlinearity $f : X \times W \to X$, where W is a Banach space.
This system we call a semilinear boundary control system (semilinear BCS).

Theorem (Proof similar to linear case (see Schwenninger 2020 & M., Prieur, 2020))

Consider the semilinear BCS with $f \in C(X \times W, X)$. Let $u \in C^2([0,\tau], U)$, and $w \in C([0,T], W)$ for some $\tau > 0$ and $x_0 \in X$ be such that $x_0 - Ru(0) \in D(A)$. Assume that the classical solution of semilinear BCS $x(\cdot)$ exists on $[0,\tau]$. Then it can be represented as

$$\begin{aligned} x(t) &= T(t)\big(x_0 - Ru(0)\big) \\ &\quad + \int_0^t T(t-r)\big(f(x(r),w(r)) + \hat{A}Ru(r) - R\dot{u}(r)\big)dr + Ru(t) \\ &= T(t)x_0 + \int_0^t T(t-r)\big(f(x(r),w(r)) + \hat{A}Ru(r)\big)dr - A\int_0^t T(t-r)Ru(r)dr \\ &= T(t)x_0 + \int_0^t T(t-r)f(x(r),w(r))dr + \int_0^t T_{-1}(t-r)(\hat{A}R - A_{-1}R)u(r)dr, \end{aligned}$$

where A_{-1} and T_{-1} are the extensions of A and T to X_{-1}. Furthermore, $A_{-1}R \in L(U, X_{-1})$ (and thus $\hat{A}R - A_{-1}R \in L(U, X_{-1})$).

Mild solutions for semilinear BCS

Definition

Let $(\hat{A}, \hat{R}, f)$ be a semilinear boundary control system with corresponding A, R. Let $x_0 \in X$, $T > 0$, $w \in L^1_{loc}([0,T], W)$, and $u \in L^1_{loc}([0,T], U)$. A continuous function $x : [0,T] \to X$ is called mild solution to the semilinear BCS on $[0,T]$ if $x(t) \in X$ for all $t > 0$ and x solves

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)\big(f(x(s), w(s)) + Bu(s)\big)ds,$$

for all $t \in [0,T]$ and where $B = \hat{A}R - A_{-1}R$. A function $x : [0,\infty) \to X$ is called a global mild solution if $x|_{[0,T]}$ is a mild solution on $[0,T]$ for all $T > 0$.

Moral: All machinery developed previously, works for semilinear BCS.

Take-out message III: Mild solutions for semilinear BCS

Mild solutions of

$$\begin{aligned}\dot{x}(t) &= \hat{A}x(t) + f(x(t), w(t)), \ t > 0,\\ \hat{R}x(t) &= u(t), \ t > 0,\end{aligned}$$

are precisely mild solutions of

$$\dot{x}(t) \quad = \quad Ax(t) + f(x(t), w(t)) + Bu(t), \qquad t > 0,$$

with

$$B = \hat{A}R - A_{-1}R.$$

Take-out message III: Mild solutions for semilinear BCS

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$$\dot{x}(t) \quad = \quad Ax(t) + f(x(t), w(t)) + Bu(t), \qquad t > 0,$$

with

$$B = \hat{A}R - A_{-1}R.$$

The theory developed previously for semilinear evolution equations, works for semilinear BCS.

Conclusions and outlook

$$\dot{x}(t) \quad = \quad Ax(t) + f(x(t), u(t)) + Bu(t), \qquad t > 0,$$

Conclusions

- Admissibility of B and Lipschitz continuity of f imply well-posedness
- Same assumptions imply certain regularity properties
- Semilinear boundary control systems can be represented as semilinear evolution equations
- This allows invoking of rich theory for ISS analysis of semilinear systems.

Conclusions and outlook

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)) + Bu(t), \quad t > 0,$$

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Conclusions and outlook

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)) + Bu(t), \quad t > 0,$$

We assumed that $f : X \times U \to X$ is a well-defined map on the whole $X \times U$.
This seemingly harmless condition is in fact quite restrictive, as $f : x \mapsto x^2$ is not well-defined on $X = L^2(0, 1)$.

What I could tell

- Well-posedness theory for analytic SEE with $f : X_\alpha \times U \to X$, $\alpha \in [0, 1)$.
- Burgers' equation with boundary L^∞-inputs and $x(0) \in H_0^1(0, \pi)$:

$$\begin{aligned} x_t &= x_{zz} - x x_z + f(z, x(z,t)) + u(z,t), \quad z \in (0, \pi), \ t > 0, \\ x(0,t) &= d(t), \quad t > 0, \\ x(\pi, t) &= 0. \end{aligned}$$

Conclusions and outlook

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)) + Bu(t), \quad t > 0,$$

What I would like to tell

- Develop converse ISS Lyapunov theorems for (non-)linear SEE/BCS.
- E.g., we still do not know, whether there is a coercive ISS Lyapunov function for

$$x_t(z,t) = x_{zz}(z,t), \quad z \in (0,1), \; t > 0,$$
$$x(0,t) = 0, \quad x(1,t) = u(t), \quad t > 0.$$

- Provide constructions of ISS Lyapunov functions for linear systems and see how that would work for semilinear ones.
- What about ISS of a viscous Burgers' equation with Dirichlet/Neumann boundary inputs?

Thank You for Your attention!