Revisiting stability of positive linear discrete-time systems

Jochen Glück, **Andrii Mironchenko**

Faculty of Mathematics and Computer Science
University of Passau

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Journal paper:
Glück, Mironchenko. *Stability criteria for positive linear discrete-time systems.* Positivity, 2021.

www.mironchenko.com

Linear discrete-time systems in Banach spaces

Let X be a Banach space, and $T \in \mathcal{L}(X)$. Consider the discrete-time system

$$x(k+1) = Tx(k), \quad k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}. \tag{1}$$

Definition

(1) is called (uniformly) exponentially stable, if: $\exists a \in [0, 1)$, $M > 0$:

$$\left\| T^k x \right\| \leq M a^k \left\| x \right\|, \quad x \in X, \quad k \in \mathbb{Z}_+.$$

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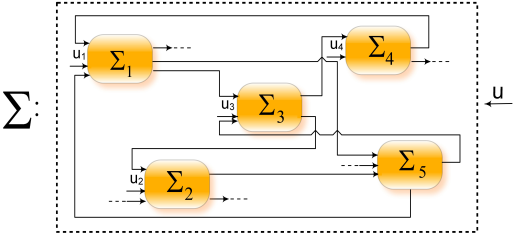
Define $r(T) := \max_{\lambda \in \sigma(T)} |\lambda|$.

Proposition (Basic (and classic) criterion for exponential stability)

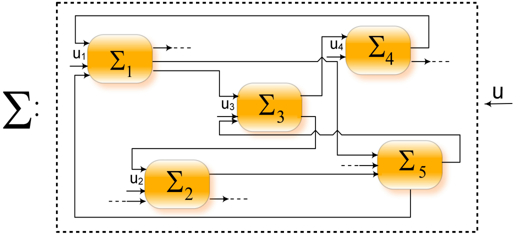
(1) is uniformly exponentially stable $\Leftrightarrow$ $r(T) < 1$.

For many more criteria see a survey part of

- Glück, AM. *Stability criteria for positive linear discrete-time systems.* Positivity, 2021.



Under which conditions is
an (in)finite interconnection of nonlinear stable systems stable?



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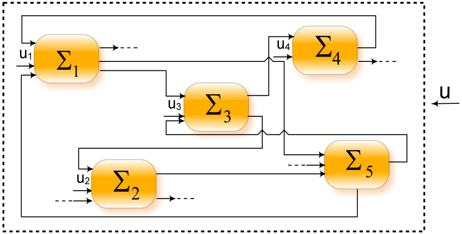
Small-gain approach

- Associate $\gamma_{ij} \geq 0$ with the influence of the j-th system onto the i-th system
- Define a gain operator $\Gamma := (\gamma_{ij})_{i,j=1}^{\infty} : \ell_p \to \ell_p, \quad p \geq 1.$
- Then $r(\Gamma) < 1$ guarantees the stability of the whole network.
- Γ is a positive operator!

Σ:



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Can we use positivity to derive more powerful criteria for uniform exponential stability?

Ordered Banach spaces

Define

$$\alpha X^+ + \beta X^+ := \{\alpha x + \beta y : x, y \in X^+\}.$$

Definition (Ordered Banach space)

An ordered Banach space is a pair (X, X^+) where X is a real Banach space, and $X^+ \subseteq X$ is a non-empty closed set such that

- $\alpha X^+ + \beta X^+ \subset X^+ \quad \forall \alpha, \beta \geq 0.$
- $X^+ \cap (-X^+) = \{0\}.$

The set X^+ is called a positive cone in X.

Having a positive cone, one can introduce a partial order on X: for $x, y \in X$

$$x \leq y \quad \Leftrightarrow \quad y - x \in X^+.$$

Generating and normal cones

Definition (Systematics of cones)

A cone X^+ in an ordered Banach space X is called:

- total if $\quad X^+ - X^+ = \{x - y : x, y \in X^+\} \quad$ is dense in X.
- generating if $X^+ - X^+ = X$.
- having nonempty interior if $\mathrm{int}\,(X^+) \neq \emptyset$.
- normal if there is $C > 0$:

$$\|x\| \leq C\,\|y\| \qquad \text{whenever } 0 \leq x \leq y.$$

Definition (Positive maps)

Let (X, X^+) be an ordered Banach space. A mapping $A : X \to X$ is called positive if $AX^+ \subseteq X^+$.

Examples of ordered Banach spaces

Example (Ordered Euclidean space)

- $(\mathbb{R}^n, \mathbb{R}^n_+)$ is an ordered Banach space with the order $x \le y \iff x_i \le y_i$, $i = 1, \ldots, n$.

Example (Ordered sequence spaces)

Let

- $X = \ell_p := \{x = (x_n) \in \mathbb{R}^{\mathbb{N}} : \|x\|_{\ell_p} < \infty\}, \quad p \in [1, \infty]$.
- $\|x\|_{\ell_p} := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ for $p < \infty$ and $\|x\|_{\ell_\infty} := \sup_{n=1}^{\infty} |x_n|$.
- $\ell_p^+ := \{(x_n)_{n \in \mathbb{Z}_+} \in \ell_p : \quad x_n \ge 0 \ \ \forall n \in \mathbb{N}\}$.

Then:

- (ℓ_p, ℓ_p^+) is an ordered Banach space
- ℓ_p^+ is generating and normal.
- $p = \infty \quad \Rightarrow \quad \operatorname{int}(\ell_\infty^+) \ne \emptyset$.

$$x(k+1) = Tx(k), \quad k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}.$$

Proposition (Well-known criteria for exponential stability of linear discrete-time systems)

Let $T \in \mathbb{R}^{n \times n}_+$. Then the following statements are equivalent:

(i) *Spectral small-gain condition:* $r(T) < 1$.

(ii) *Small-gain condition:* $Tx \not\geq x \quad \forall x \in \mathbb{R}^n_+, x \neq 0$.

(iii) *There is a point of strict decay:* There are $\lambda \in (0,1)$ and $x \in \operatorname{int}(\mathbb{R}^n_+)$: $Tx \leq \lambda x$.

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Proof.

(ii) $\Rightarrow$ (i). Assume that (i) doesn't hold, that is, $r(T) \geq 1$. By Perron-Frobenius theorem, there is $x \in \mathbb{R}^n_+$, $x \neq 0$ so that $Tx = r(T)x \geq x$, which contradicts to (ii). □

What about infinite-dimensions?

Proposition (Well-known criteria for exponential stability of linear discrete-time systems)

Let $T \in \mathbb{R}_+^{n\times n}$. Then the following statements are equivalent:

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(ii) Small-gain condition: $Tx \not\geq x \quad \forall x \in X^+, x \neq 0$.

(iii) There is a point of strict decay: There are $\lambda \in (0,1)$ *and* $x \in \mathrm{int}\,(X^+)$: $Tx \leq \lambda x$.

Can we extend this result to positive operators in an ordered Banach space (X, X^+)?

- If $\mathrm{int}\,(X^+) = \emptyset$, the condition (iii) is never satisfied.
- There is an infinite-dimensional version of Perron-Frobenius theorem: Krein-Rutman theorem.
- However, it requires that the operator T is quasi-compact, which is a rather strong assumption.

Example

Example (Discrete-time system, induced by right shift)

$$x(k+1) = 2Rx(k), \quad k \in \mathbb{Z}_+,$$

- $X := (\ell_\infty, \ell_\infty^+)$.
- The cone ℓ_∞^+ is normal and has non-empty interior.
- R is the right shift on X, i.e.,

$$R(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots).$$

Clearly, the small-gain condition holds:

$$2Rx \not\geq x \qquad \text{for all } x \in X^+ \setminus \{0\},$$

At the same time, $\|(2R)^k x\| \to \infty$ as $k \to \infty$ for each $x \neq 0$.

Moral

$Tx \not\geq x \ \ \forall x \in X^+ \backslash \{0\}$ is too weak for $r(T) < 1$.

How can we fix $Tx \not\geq x$?

The following statements are clearly equivalent:

- $Tx \not\geq x \quad \forall x \in X^+ \backslash \{0\}$
- $Tx - x = (T - \mathrm{id})x \notin X^+ \quad \forall x \in X^+ \backslash \{0\}$
- $\mathrm{dist}\left((T - \mathrm{id})x, X^+\right) > 0 \quad \forall x \in X^+ \backslash \{0\}$

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Lack of uniformity

However: It still may happen that

$$\inf_{\|x\|=1,\ x \in X^+} \mathrm{dist}\left((T - \mathrm{id})x, X^+\right) = 0.$$

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Uniform small-gain condition

$$\eta := \inf_{\|x\|=1,\ x \in X^+} \operatorname{dist}\big((T - \mathrm{id})x, X^+\big) > 0$$

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Uniform small-gain condition (equivalent form)

There is $\eta > 0$ such that

$$\mathrm{dist}\left((T - \mathrm{id})x, X^+\right) \geq \eta \|x\|, \quad x \in X^+.$$

General stability criteria for positive systems

Uniform small-gain condition

There is a number $\eta > 0$ such that

$$\operatorname{dist}\big((T - \mathrm{id})x, X^+\big) \geq \eta \, \|x\| \,, \quad x \in X^+.$$

Theorem (AM, Glück, Positivity, 2021)

Let (X, X^+) be an ordered Banach space with generating and normal cone and let $T \in \mathcal{L}(X)$ be positive. Then

$$r(T) < 1 \quad \Leftrightarrow \quad T \text{ satisfies the uniform small-gain condition.}$$

Proof.

See the paper

- Glück, AM. *Stability criteria for positive linear discrete-time systems*, Positivity, 2021,

where many more equivalent conditions are shown. □

Cones with non-empty interior

Theorem (AM, Glück, Positivity, 2021)

Let

- (X, X^+) *be an ordered Banach space*
- X^+ *is normal and has non-empty interior*
- $T \in \mathcal{L}(X)$ *is positive.*

The following statements are equivalent:

(i) *Spectral small-gain condition:* $r(T) < 1$.

(ii) *Exponential stability:* $\exists\ M > 0$, $a \in (0,1)$:

$$\|T^k x\|_X \leq Ma^k \|x\|_X, \quad x \in X^+, \quad k \in \mathbb{Z}_+.$$

(iii) *There is a point of strict decay:* $\exists\ z \in \mathrm{int}\,(X^+)$ *and* $\lambda \in (0,1)$: $Tz \leq \lambda z$.

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Remark

For applications it is important to have an explicit expression of z in the item (iii).

Constructive proof of (ii) $\Rightarrow$ (iii).

- Pick any $\lambda \in (a, 1)$, any $y \in \operatorname{int}(X^+)$, and consider the vector

$$z_N := \sum_{k=0}^{N} \frac{T^k(y)}{\lambda^{k+1}}, \quad N \in \mathbb{N} \cup \{+\infty\}.$$

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Note that $z_{+\infty} = \sum_{k=0}^{\infty} \frac{T^k(y)}{\lambda^{k+1}} = (\lambda \,\mathrm{id} - T)^{-1} y$.

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- As T is positive, $z_N \geq \frac{1}{\lambda} y \ \ \forall N \in \mathbb{N} \cup \{+\infty\}$, implying $z_N \in \mathrm{int}\,(X^+)$.

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- Now decompose $T(z_N)$ as

$$T(z_N) = \big(T - \lambda \operatorname{id} + \lambda \operatorname{id}\big)(z_N) = T\Big(\sum_{k=0}^{N} \frac{T^k(y)}{\lambda^{k+1}}\Big) - \sum_{k=0}^{N} \frac{T^k(y)}{\lambda^k} + \lambda z_N.$$

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  Via linearity, we obtain for large enough N:

$$T(z_N) = \sum_{k=0}^{N} T\Big(\frac{T^k(y)}{\lambda^{k+1}}\Big) - \sum_{k=0}^{N} \frac{T^k(y)}{\lambda^k} + \lambda z_N = \sum_{k=0}^{N} \frac{T^{k+1}(y)}{\lambda^{k+1}} - \sum_{k=0}^{N} \frac{T^k(y)}{\lambda^k} + \lambda z_N$$

$$= -y + \frac{T^{N+1}(y)}{\lambda^{N+1}} + \lambda z_N \leq \lambda z_N.$$

Quasi-compact operators

Theorem (AM, Glück, Positivity, 2021)

Let (X, X^+) be an ordered Banach space with total cone and let $T \in \mathcal{L}(X)$ be positive and quasi-compact. Then:

$$r(T) < 1 \quad \Leftrightarrow \quad Tx \not\geq x \quad \forall x \in X^+ \backslash \{0\}.$$

Proof.

Follows by invocation of Krein-Rutman theorem. See many more characterizations for $r(T) < 1$ in the paper. □

Conclusions

Characterizations for $r(T) < 1$ for positive $T \in L(X)$ in (X, X^+).

- If X^+ is normal and generating, then:
$$r(T) < 1 \quad \Leftrightarrow \quad \exists \eta > 0\colon \quad \mathrm{dist}\left((T - \mathrm{id})x, X^+\right) \geq \eta \left\|x\right\|, \quad x \in X^+.$$
- If X^+ is normal and having interior points, then:
$$r(T) < 1 \quad \Leftrightarrow \quad \exists\, z \in \mathrm{int}\,(X^+) \text{ and } \lambda \in (0,1)\colon \quad Tz \leq \lambda z.$$
- If T is quasi-compact, and X^+ is total, then:
$$r(T) < 1 \quad \Leftrightarrow \quad Tx \not\geq x \quad \forall x \in X^+ \backslash \{0\}.$$

What I could tell

- Many more characterizations of $r(T) < 1$ in each of above cases
- In the journal version one can find also a survey showing the position of our results in the state of the art in stability of linear discrete-time systems.

Main Reference

- Glück, AM. *Stability criteria for positive linear discrete-time systems*, Positivity, 2021.

Outlook & Extensions

Nonlinear monotone discrete-time systems & Stability of infinite networks

- AM, Noroozi, Kawan, Zamani. *ISS small-gain criteria for infinite networks with linear gain functions*, SCL, 2021.
- AM, Kawan, Glück. *Nonlinear small-gain theorems for input-to-state stability of infinite interconnections*, MCSS, 2021.
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Extensions to linear positive continuous-time systems

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Thank you for Your attention!

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Do not miss my talk
'Small-gain conditions for robust stability of nonlinear infinite networks'
Tuesday, 11:20–11:55, session 'Dissipativity Theory II: Stability Analysis'.