Lyapunov methods for robust stability of distributed parameter systems

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Outline

Class of systems

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X, \\ x(0) = \phi_0. \end{cases}$$

- $U_c = C(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.
- $f(0,0) = 0$.

$x \in C([0,T], X)$ is a mild solution iff

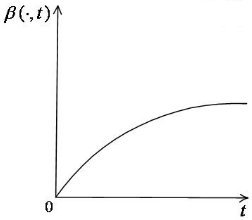
$$x(t) = T(t)\phi_0 + \int_0^t T(t-s)f(x(s), u(s))ds.$$

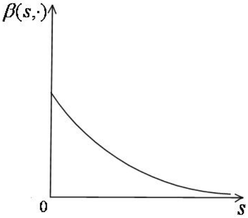
Comparison functions

$$\mathcal{K}_\infty := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma(0) = 0,\ \gamma \text{ is continuous, increasing and unbounded}\}$$

$$\mathcal{L} := \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly decreasing and } \lim_{t\to\infty} \gamma(t) = 0\right\}$$

$$\mathcal{KL} := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid\ \beta(\cdot, t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r, \cdot) \in \mathcal{L},\ \forall r > 0\}$$





Input-to-state stability

Definition (GAS uniform w.r.t. state (0-UGASx))

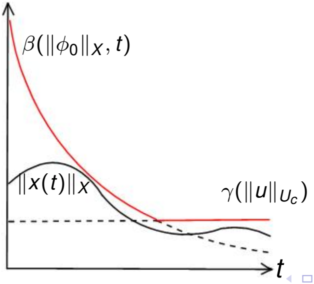
0-UGASx $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}$: $\forall \phi_0 \in X, \forall t \geq 0$

$$\|\phi(t, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t).$$

Definition (ISS)

ISS $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$: $\forall t \geq 0, \forall \phi_0 \in X, \forall u \in U_c$

$$\|\phi(t, \phi_0, u)\|_X \leq \max\Big\{\beta(\|\phi_0\|_X, t), \underbrace{\gamma}_{Gain}\Big(\sup_{s\in[0,t]} \|u(s)\|_U\Big)\Big\}.$$



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Definition (integral input-to-state stability (iISS))

iISS $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}, \alpha, \mu \in \mathcal{K}_\infty$: $\forall t \geq 0, \forall \phi_0 \in X, \forall u \in U_c$

$$\alpha(\|\phi(t, \phi_0, u)\|_X) \leq \beta(\|\phi_0\|_X, t) + \int_0^t \mu(\|u(s)\|_U) ds.$$

Motivation

Applications of ISS/iISS

1. Unified theory of internal and external stability
2. Robust stabilization of nonlinear systems
3. Design of robust nonlinear observers
4. Nonlinear detectability
5. ISS feedback redesign
6. Stability of nonlinear networked control systems
7. ...

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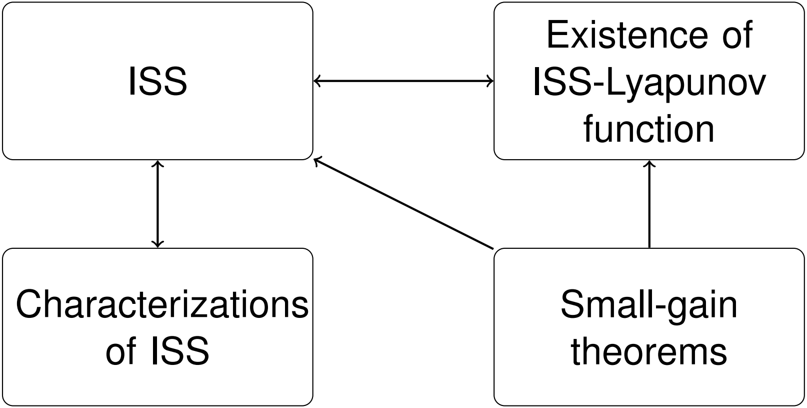
ISS/iISS framework has been developed for

1. ODEs
2. Time-delay systems
3. Discrete-time systems
4. Hybrid and impulsive systems

Fundamentals of ISS-Theory for ODEs



For linear systems ISS = iISS

$$\Sigma: \quad \dot{x} = Ax + Bu, \quad x(0) = x_0, \quad B \in L(U, X).$$

0-GAS := Σ is globally attractive $\quad \Leftrightarrow \quad \lim_{t\to\infty} \|T(t)x\| = 0 \ \forall x \in X$.

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Proposition

Σ is ISS $\Leftrightarrow$ Σ is iISS $\Leftrightarrow$ Σ is 0-UGASx $\underset{\not\Leftarrow}{\Rightarrow}$ Σ is 0-GAS

Lyapunov functions

$$\dot{x}(t) = Ax(t) + Bu(t) + C(x(t), u(t)),$$
$$x(0) = \phi_0,$$

Definition

$V : X \to \mathbb{R}_+$ is an iISS-Lyapunov function iff $\exists \psi_1, \psi_2, \sigma, \alpha \in \mathcal{K}$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $\dot{V}_u(x) \leq -\alpha(V(x)) + \sigma(\|u(0)\|_U)$.

If $\alpha \in \mathcal{K}_\infty$, then V is called an ISS-Lyapunov function.

Theorem

$\exists$ *ISS/iISS Lyapunov function* $\Rightarrow$ *ISS/iISS.*

Bilinear systems

$$\dot{x}(t) = Ax(t) + Bu(t) + C(x(t), u(t)),$$

where $B \in L(U, X)$, and $\exists K > 0$:

$$\|C(x,u)\|_X \leq K\|x\|_X\|u\|_U.$$

Typical example:

$$\dot{x} = -x + xu.$$

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For $V(x) = \ln(1 + x^2)$ we have

$$\dot{V}(x) = \frac{2x\dot{x}}{1 + x^2} = -\frac{2x^2}{1 + x^2} + \frac{2x^2 u}{1 + x^2} \leq -\frac{2x^2}{1 + x^2} + 2u = -(1 - e^{-V(x)}) + 2u.$$

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Theorem (E. Sontag, 1998)

Finite-dimensional bilinear 0-GAS systems are iISS.

Bilinear systems

$$\dot{x}(t) = Ax(t) + Bu(t) + C(x(t), u(t)). \qquad \text{(BL)}$$

Theorem (A.M., H. Ito, submitted to CDC 2014)

Let X be a Hilbert space, A generate an analytic semigroup on X, and let there exist a coercive positive self-adjoint operator $P \in L(X)$:

$$\langle Ax, Px \rangle + \langle Px, Ax \rangle = -\|x\|_X^2, \quad \forall x \in D(A).$$

Then (BL) *is iISS with an iISS Lyapunov function*

$$W(x) = \ln\Big(1 + \langle Px, x \rangle\Big).$$

Proof.

1. We prove that W is an ISS-Lyapunov function on $D(A)$
2. Use density argument to prove this on X.

Bilinear systems

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(BL) *is iISS* $\Leftrightarrow$ (BL) *is 0-UGASs.*

Choice of state and input space

How do we understand bilinearity:

$$\|C(x,u)\|_X \leq K\|x\|_X\|u\|_U.$$

Consider

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial l^2}(l,t) + x(l,t)u(l,t),\ (l,t) \in (0,1) \times (0,\infty)$$
$$x(0,t) = x(1,t) = 0.$$

If $X = L_p(0,1)$:

- The choice $U = L_q(0,1)$ is not possible for $q < \infty$.
- One can choose $U = C(0,1)$.

Example of nonexponentially ISS system

$$\begin{cases} \frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial l^2} + ax - x\left(\frac{\partial x}{\partial l}\right)^2 + u, \quad \forall t > 0, \\ x(0,t) = x(\pi,t) = 0. \end{cases}$$

- $X = H_0^1(0,\pi)$.

Theorem (A.M.)

1. *If* $U = W_0^{1,\frac{4}{3}}(0,\pi) \Rightarrow$ *ISS iff* $a \leq 1$.
2. *If* $U = L_2(0,\pi) \Rightarrow$ *ISS provided* $a < 1$.

$$V(x) = \int_0^{\pi} \left(\frac{\partial x}{\partial l}\right)^2 dl = \|x\|^2_{H_0^1(0,\pi)}.$$

The derivative of V along the trajectories is equal

$$\begin{aligned}\dot{V}(x) \quad = \quad & -2\int_0^{\pi} \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl + 2a\int_0^{\pi} \left(\frac{\partial x}{\partial l}\right)^2 dl \\ & -\frac{2}{3}\int_0^{\pi} \left(\frac{\partial x}{\partial l}\right)^4 dl - 2\int_0^{\pi} \frac{\partial^2 x}{\partial l^2} u dl.\end{aligned}$$

$$V(x) = \int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl = \|x\|^2_{H_0^1(0,\pi)}.$$

The derivative of V along the trajectories is equal

$$\begin{aligned}\dot{V}(x) &= -2\int_0^\pi \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl + 2a\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl \\ &\quad -\frac{2}{3}\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^4 dl - 2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u\, dl.\end{aligned}$$

$$U = L_2(0,\pi) \quad \Rightarrow \quad -2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u\, dl \le 2\omega \int_0^\pi \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl + 2\frac{1}{\omega}\int_0^\pi u^2 dl$$

$$V(x) = \int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl = \|x\|^2_{H^1_0(0,\pi)}.$$

The derivative of V along the trajectories is equal

$$\begin{aligned}\dot V(x) &= -2\int_0^\pi \left(\frac{\partial^2 x}{\partial l^2}\right)^2 dl + 2a\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^2 dl \\ &\quad -\frac{2}{3}\int_0^\pi \left(\frac{\partial x}{\partial l}\right)^4 dl - 2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u\, dl.\end{aligned}$$

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$$U = W_0^{1,\frac{4}{3}}(0,\pi) \quad \Rightarrow \quad -2\int_0^\pi \frac{\partial^2 x}{\partial l^2} u\, dl \le \frac{\omega}{4}\int_0^\pi \left|\frac{\partial x}{\partial l}\right|^4 dl + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{4}\int_0^\pi \left|\frac{\partial u}{\partial l}\right|^{\frac{4}{3}} dl$$

Case $U = L_2(0,\pi)$

$$\dot{V}(x) \leq -2(1-a-\omega)V(x) - \frac{2}{3\pi}V^2(x) + 2\frac{1}{\omega}\int_0^\pi u^2 dl.$$

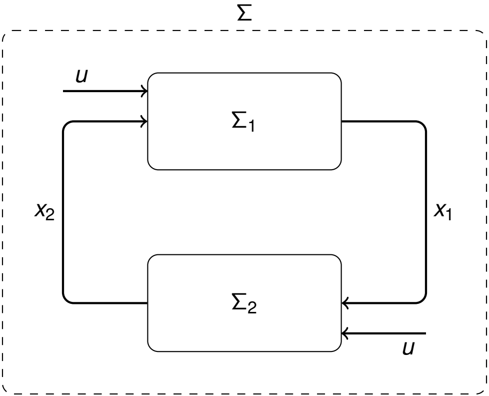
For $a < 1$ one can take $\omega < 1 - a \Rightarrow$ ISS for $U = L_2(0,\pi)$ and $a < 1$.

Case $U = W_0^{1,\frac{4}{3}}(0,\pi)$

$$\dot{V}(x) \leq -2(1-a)V(x) - \left(\frac{1}{3} - \frac{\omega}{2}\right)\frac{1}{\pi}V^2(x) + \frac{1}{\omega^{\frac{1}{3}}}\frac{3}{2}\|u\|_{W_0^{1,\frac{4}{3}}(0,\pi)}.$$

ISS for $U = L_2(0,\pi)$ and $a \leq 1$

Interconnections of 2 systems



$$gain_{\Sigma_1 \to \Sigma_2} \circ gain_{\Sigma_2 \to \Sigma_1}(s) < s \ \Rightarrow \text{iISS of } \Sigma$$

Interconnections of 2 systems

$$\Sigma : \begin{cases} \Sigma_1 : & \dot{x}_1 = A_1 x_1 + f_1(x_1, x_2, u), \ \ x_1 \in X_1 \\ \Sigma_2 : & \dot{x}_2 = A_2 x_2 + f_2(x_1, x_2, u), \ \ x_2 \in X_2 \end{cases}$$

ISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is ISS-Lyapunov functions for Σ_i, $i = 1, 2$ iff

- $V_1(x_1) \geq \max\left\{\chi_{12}(V_2(x_2)), \chi_1(\|u(0)\|_U)\right\} \ \Rightarrow \ \dot{V}_1(x_1) \leq -\alpha_1(V_1(x_1))$,
- $V_2(x_2) \geq \max\left\{\chi_{21}(V_1(x_1)), \chi_2(\|u(0)\|_U)\right\} \ \Rightarrow \ \dot{V}_2(x_2) \leq -\alpha_2(V_2(x_2))$,

Small-gain theorem for 2 interconnected systems

Theorem (Dashkovskiy, M., MCSS, 2013)

Let V_1, V_2 be ISS-Lyapunov function for Σ_1, Σ_2 with gains χ_{12}, χ_{21}.

$$\chi_{12} \circ \chi_{21} < id \qquad \text{(SGB)}$$

$\Rightarrow$

- Σ *is ISS.*
- $V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}$ *is a Lyapunov function for* Σ.

Example: interconnected diffusion equations

$$\begin{cases} \frac{\partial x_1}{\partial t} = q_1 \frac{\partial^2 x_1}{\partial z^2} + x_2^2, \quad z \in (0,\pi), \\ x_1(0,t) = x_1(\pi,t) = 0; \\ \frac{\partial x_2}{\partial t} = q_2 \frac{\partial^2 x_2}{\partial z^2} + \sqrt{|x_1|}, \quad z \in (0,\pi), \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases} \tag{1}$$

State spaces:

$$X_1 = X_2 = L_2(0,d)$$

Generators:

$$A_i = c_i \frac{d^2}{dx^2} \quad \text{with} \quad D(A_i) = H_0^1(0,d) \cap H^2(0,d).$$

ISS Lyapunov functions:

$$V_1(x_1) = \int_0^\pi x_1^2(z)dz = \|x_1\|^2_{L_2(0,\pi)},$$

$$V_2(x_2) = \int_0^\pi x_2^4(z)dz = \|x_2\|^4_{L_4(0,\pi)}.$$

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$$V_2(x_2) = \int_0^\pi x_2^4(z)dz = \|x_2\|^4_{L_4(0,\pi)}.$$

Lie derivative of V_1:

$$\begin{aligned}\frac{d}{dt}V_1(x_1) &= 2\int_0^\pi x_1(z,t)\left(q_1\frac{\partial^2 x_1}{\partial z^2}(z,t) + x_2^2(z,t)\right)dz \\ &\leq -2q_1\left\|\frac{dx_1}{dz}\right\|^2_{L_2(0,\pi)} + 2\|x_1\|_{L_2(0,\pi)}\|x_2\|^2_{L_4(0,\pi)}.\end{aligned}$$

By Friedrichs' inequality

$$\begin{aligned}\frac{d}{dt}V_1(x_1) &\leq -2q_1\|x_1\|^2_{L_2(0,\pi)} + 2\|x_1\|_{L_2(0,\pi)}\|x_2\|^2_{L_4(0,\pi)} \\ &= -2q_1V_1(x_1) + 2\sqrt{V_1(x_1)}\sqrt{V_2(x_2)}.\end{aligned}$$

Take

$$\chi_{12}(r) = \frac{1}{q_1^2(1-\varepsilon)^2} r, \ \forall r > 0,$$

with arbitrary $a > 0$. We obtain

$$V_1(x_1) \geq \chi_{12}(V_2(x_2)) \quad \Rightarrow \quad \frac{d}{dt} V_1(x_1) \leq -2\varepsilon q_1 V_1(x_1).$$

Analogously for

$$\chi_{21}(r) = \Big(\frac{3}{4} q_2 (1-\varepsilon)\Big)^{-4} r, \ \forall r > 0,$$

we have

$$V_2(x_2) \geq \chi_{21}(V_1(x_1)) \quad \Rightarrow \quad \frac{d}{dt} V_2(x_2) \leq -3\varepsilon q_2 V_2(x_2).$$

$$\chi_{12} \circ \chi_{21} < \mathrm{Id} \quad \Leftrightarrow \quad q_1^2 \Big(\frac{3}{4} q_2\Big)^4 > 1.$$

Interconnected systems

$$\Sigma : \begin{cases} \Sigma_i : \quad \dot{x}_i = A_i x_i + f_i(x_1, \ldots, x_n, u), \;\; x_i \in X_i \\ i = 1, \ldots, n \end{cases}$$

- X_i state space of Σ_i
- A_i infinitesimal generator of C_0-semigroup on X_i.
- $X = X_1 \times \ldots \times X_n$ state space of the whole system.
- $\tilde{X}_i := X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n \times U$
  space of inputs into i-th subsystem.

ISS-Lyapunov functions for subsystems

$$\Sigma : \begin{cases} \Sigma_i : \quad \dot{x}_i = A_i x_i + f_i(x_1, \ldots, x_n, u), \; x_i \in X_i \\ i = 1, \ldots, n \end{cases}$$

ISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is ISS-Lyapunov function for Σ_i iff
$\exists \psi_{i1}, \, \psi_{i2}, \, \alpha_i, \, \chi_i, \, \chi_{ij} \in \mathcal{K}_\infty, \quad j = 1, \ldots, n$:

- $\psi_{i1}(\|x_i\|_{X_i}) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|_{X_i})$
- $V_i(x_i) \geq \max\left\{ \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U) \right\} \;\Rightarrow\; \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)),$

$\forall x_i \in X_i, \; \forall \tilde{x}_i \in \tilde{X}_i, \forall v \in PC(\mathbb{R}_+, \tilde{X}_i)$ with $v(0) = \tilde{x}_i$.

Small-gain theorem

Gain matrix: $\Gamma_M = (\chi_{ij})_{i,j=1,\ldots,n}$, $\chi_{ij} \in \mathcal{K}_\infty \cup \{0\}$.
Gain operator: $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$

$$\Gamma(s) := \left(\max_{j=1}^{n} \chi_{1j}(s_j), \ldots, \max_{j=1}^{n} \chi_{nj}(s_j) \right), \; s \in \mathbb{R}^n_+.$$

Theorem (Dashkovskiy, M., MCSS, 2013)

Let V_i be ISS-Lyapunov function for Σ_i with gains χ_{ij}.

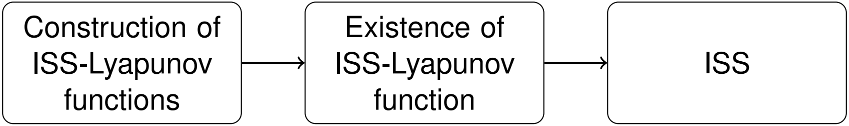
$$\Gamma(s) \not\geq s, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\} \qquad \text{(SGB)}$$

$\Rightarrow$

- *Σ is ISS.*
- *$V(x) := \max_i\{\sigma_i^{-1}(V_i(x_i))\}$ is a Lyapunov function for Σ.*

For $n = 2$ (SGB) $\Leftrightarrow$ $\chi_{12} \circ \chi_{21} < id$.

Summary and Outlook



Other results in ISS theory

- Linearization method for study of LISS
- ISS of infinite-dimensional impulsive systems
    - Dwell-time conditions for ISS of impulsive systems
    - Small gain theorems for impulsive systems
    - Small gain theorems for time-delay systems

Plans for the near future

- ISS of linear systems $\dot{x}(t) = Ax(t) + Cu(t)$, with unbounded C.
- Converse ISS Lyapunov theorem
- Characterisation of ISS for ∞-dim systems.
- Applications of ISS Theory
- Robust Stabilisation of PDEs
- "Integral ISS" theory for ∞-dim systems.

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Thank you for attention!