Lyapunov functions for boundary control systems

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B. Jacob, A. Mironchenko, J. Partington and F. Wirth. Remarks on input-to-state stability and non-coercive Lyapunov functions, CDC 2018.

Example: stabilization of a linear heat equation

$$(\Sigma_1) \qquad \begin{array}{l} \frac{\partial x}{\partial t}(z,t) = \frac{\partial^2 x}{\partial z^2}(z,t) + ax(z,t) \\ x(0,t) = 0 \quad \forall t \geq 0 \\ x(1,t) = u(t) + d(t) \quad \forall t \geq 0. \end{array}$$

If $a > \pi$, then this system is unstable, for $u + d \equiv 0$.

Aim

To design a feedback controller

$$u(t) = p(x(\cdot, t))$$

which uniformly globally asymptotically stabilizes the system, robustly w.r.t. actuator disturbances d.

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$$(\Sigma_1) \quad \begin{array}{l} \frac{\partial x}{\partial t}(z,t) = \frac{\partial^2 x}{\partial z^2}(z,t) + ax(z,t) \\ x(0,t) = 0 \quad \forall t \geq 0 \\ x(1,t) = u(t) + d(t) \quad \forall t \geq 0. \end{array} \quad \xrightarrow[\text{Volterra tr.}]{} \quad \begin{array}{l} \frac{\partial w}{\partial t}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) \\ w(0,t) = 0 \quad \forall t \geq 0 \\ w(1,t) = d(t) \quad \forall t \geq 0, \end{array} \quad (\Sigma_2)$$

Σ_1 is transformed into Σ_2 by means of

- $w(z,t) = x(z,t) + \int_0^z k(z,y)x(y,t)dy$
- $u(t) = -\int_0^1 k(1,y)x(y,t)dy$
- k is a kernel of a Volterra transformation, which can be computed explicitly.

and we naturally come to the question of ISS of Σ_2.

Questions

- How to show that Σ_2 is ISS?
- Can we derive some systematic methods for that?

Definition

The triple $\Sigma = (X, \mathcal{U}, \phi)$ is called control system, if:

(Σ1) Forward-completeness: for every $x \in X$, $u \in \mathcal{U}$ and for all $t \geq 0$ the value $\phi(t, x, u) \in X$ is well-defined.

(Σ2) Continuity: for each $(x, u) \in X \times \mathcal{U}$ the map $t \mapsto \phi(t, x, u)$ is continuous.

(Σ3) Cocycle property: for all $t, h \geq 0$, for all $x \in X$, $u \in \mathcal{U}$ we have

$$\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u).$$

Class of systems

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Examples

- Ordinary differential equations
- Evolution Partial differential equations with Lipschitz nonlinearities
- Broad classes of boundary control systems
- Time-delay systems
- Heterogeneous systems with distinct components

Example (Linear systems with admissible control operators)

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in X, t > 0. \tag{1}$$

- A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X
- $B \in L(U, X_{-1})$ for some Banach space U
- X_{-1} is the completion of X w.r.t. $\|x\|_{X_{-1}} := \|(\beta - A)^{-1}x\|_X$, for some $\beta \in \rho(A)$.
- T extends uniquely to T_{-1} on X_{-1} whose generator A_{-1} is an extension of A.
- (1) is well-posed on X_{-1}: $\forall x_0 \in X$ and $\forall u \in L^1_{\mathrm{loc}}([0,\infty), U)$, the function $x : [0,\infty) \to X_{-1}$,

$$x(t) := T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds, \quad t \geq 0,$$

is called mild solution of (1).

That's great, but the trajectory is now in X_{-1}

Example (...Continue...)

- $B \in L(U, X_{-1})$ is called an q-admissible control operator for $(T(t))_{t\geq 0}$, where $1 \leq q \leq \infty$, if $\forall t \geq 0$ and $\forall u \in L^q([0,\infty), U)$:

$$\int_0^t T_{-1}(t-s)Bu(s)ds \in X.$$

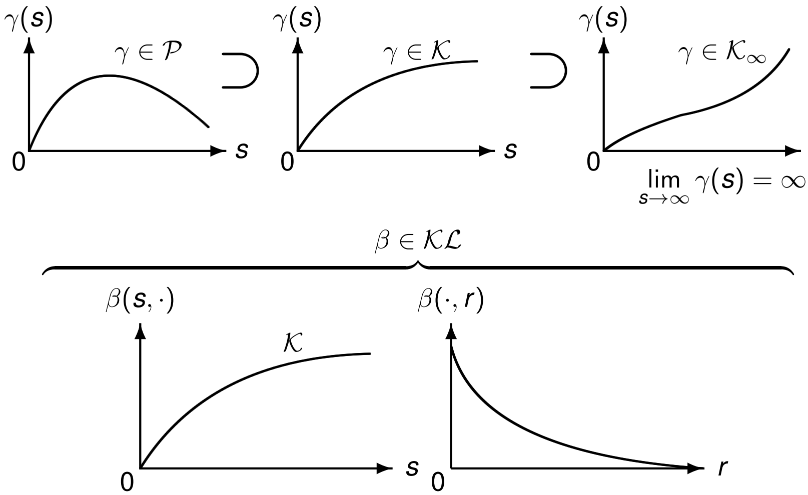
Triple $(X, L^\infty([0,\infty), U), \phi)$ defines a control system if:

- B is ∞-admissible
- $\forall x_0 \in X$, $\forall u \in L^\infty([0,\infty), U)$ the mild solution $\phi(\cdot, x_0, u)$ is continuous.

Above 2 conditions are implied by each of the following conditions:

- B is q-admissible for some $q \in [1, \infty)$
- B is ∞-admissible, $\dim U < \infty$, X is a Hilbert space and A generates an analytic semigroup which is similar to a contraction semigroup

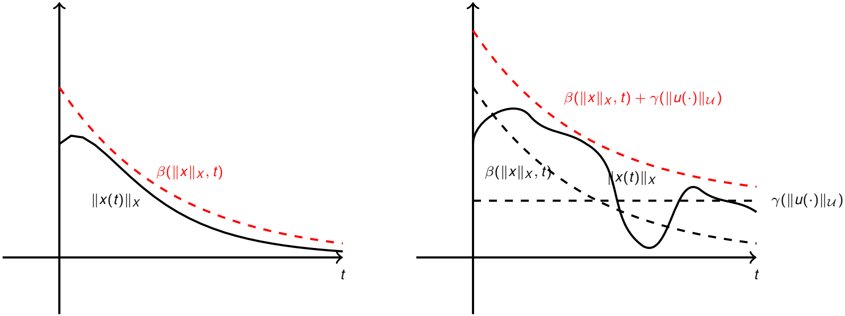
Comparison functions



Definition (E. Sontag, 1989, for ODEs)

ISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall x \in X, \forall u \in \mathcal{U}$

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}).$$



(a) $u \equiv 0$ (b) $u \not\equiv 0$

Motivation

Why ISS?

1. Unified theory of internal and external stability
   - E. D. Sontag. *Input to State Stability: Basic Concepts and Results.* In Nonlinear and Optimal Control Theory, chapter 3, 2008.
2. Robust stabilization of nonlinear systems
   - M. Krstić, I. Kanellakopoulos, P. Kokotović. Nonlinear and adaptive control design, Wiley, 1995.
3. Design of robust nonlinear observers
   - M. Arcak, P. Kokotović. Nonlinear observers: a circle criterion design and robustness analysis, 2001.
4. Stability of networks of nonlinear control systems
   - Z.-P. Jiang, I. Mareels, Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems, Automatica, 1996.
   - S. Dashkovskiy, B. Rüffer, F. Wirth. Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions, SICON, 2010.
5. ...

Motivation

Infinite-dimensional ISS theory: 2008 – now

- **Linear systems: Characterizations and applications**
  F. Bribiesca Argomedo, B. Jacob, I. Karafyllis, M. Krstic, F. Mazenc, AM, R. Nabiullin, J. R. Partington, C. Prieur, J. Schmid, F. Schwenninger, F. Wirth, E. Witrant, H. Zwart, . . .
- **Nonlinear systems: Lyapunov theory, small-gain theorems**
  M. Ahmadi, A. Chaillet, Y. Chitour, S. Dashkovskiy, G. Is. Detorakis, M. Edalatzadeh, H. Ito, B. Jayawardhana, Z.-P. Jiang, I. Karafyllis, M. Krstic, H. Logemann, S. Marx, F. Mazenc, AM, K. Morris, S. Palfi, A. Papachristodoulou, A. Pisano, C. Prieur, E. P. Ryan, S. Senova, Y. Orlov, A. Tanwani, S. Tarbouriech, G. Valmorbida, J. Zheng, G. Zhu, . . .
- **Nonlinear systems: Characterizations, non-Lyapunov methods**
  B. Jacob, I. Karafyllis, M. Krstic, AM, J. Schmid, F. Schwenninger, F. Wirth, J. Zheng, G. Zhu, . . .

Almost $2/3$ of papers appeared since 2016.

ISS Lyapunov functions

Definition

$V : X \to \mathbb{R}_+$ is a non-coercive ISS-Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \le \psi_2(\|x\|_X) \quad \forall x \neq 0$

(ii) $\dot{V}_u(x) \le -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \quad \forall x \in X, \forall u \in \mathcal{U}$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}\big(V(\phi(t,x,u)) - V(x)\big).$$

V is called a coercive ISS-Lyapunov function if

$$\exists \psi_1, \psi_2 \in \mathcal{K}_\infty : \quad \psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X), \quad \forall x \neq 0.$$

Theorem (Direct Lyapunov theorem)

$\exists$ a coercive ISS Lyapunov function $\quad \Rightarrow \quad$ ISS.

Converse Lyapunov Theorems for Lipschitz rhs

Theorem (Mironchenko, Wirth, SCL 2018)

Consider

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X,$$

- $\mathcal{U} = PC(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.
- T *is a* C_0*-semigroup over Banach space* X.
- f *is a bi-Lipschitz continuous perturbation.*

The following statements are equivalent:

(i) Σ *is ISS.*

(ii) *There exists a coercive ISS Lyapunov function for* Σ *which is locally Lipschitz continuous.*

What we have right now

- Using coercive ISS Lyapunov functions combined with integral inequalities, one can successfully study ISS of PDEs with distributed disturbances
- For ISS systems which are regular enough (bi-Lipschitz) a coercive ISS Lyapunov function always exists.
- At the same time, study of PDEs with boundary disturbances calls for new methods for ISS analysis.

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Approaches, proposed to tackle this problem include

- Novel constructions of coercive LFs
  - for systems of conservation laws
  - for reaction-diffusion equations with Neumann and Robin boundary conditions
- Monotonicity methods (for nonlinear reaction-diffusion equations with Dirichlet boundary conditions)
- Non-coercive ISS Lyapunov functions

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- Non-coercive ISS Lyapunov functions

Definition

Let $\Sigma = (X, \mathcal{U}, \phi)$ be given.

- We call $0 \in X$ an equilibrium point (of the undisturbed system) if $\phi(t, 0, 0) = 0$ for all $t \geq 0$.
- We say Σ has the CEP property, if 0 is an equilibrium and for every $\varepsilon > 0$ and for any $h > 0$ there exists a $\delta = \delta(\varepsilon, h) > 0$, so that

$$t \in [0, h], \ \|x\|_X \leq \delta, \ \|u\|_{\mathcal{U}} \leq \delta \ \Rightarrow \ \|\phi(t, x, u)\|_X \leq \varepsilon. \tag{2}$$

- Σ has bounded reachability sets (BRS), if:

$$C > 0, \ \tau > 0 \quad \Rightarrow \quad \sup_{\|x\|_X \leq C, \ \|u\|_{\mathcal{U}} \leq C, \ t \in [0,\tau]} \|\phi(t, x, u)\|_X < \infty.$$

Definition

$V : X \to \mathbb{R}_+$ is a non-coercive ISS-Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

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Why non-coercive Lyapunov functions?

- LFs for linear systems via Lyapunov equation are non-coercive.
- Very useful for boundary control systems
- May be of use for couplings of infinitely-many systems (for Lyapunov small-gain theorems)
- ...

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Non-coercive LFs are frequently used for linear systems.
Next we show an essentially nonlinear result.

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Theorem (Jacob, Mironchenko, Partington, Wirth, to be submitted, 2019)

Let Σ be a BRS control system, which is continuous at equlibrium.
If $\exists$ a non-coercive ISS Lyapunov function for $\Sigma \quad \Rightarrow \quad \Sigma$ is ISS.

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- We cannot use 'linear' methods
- We cannot use comparison principle

A proof relies deeply on characterizations of ISS, obtained in

- A.M. *Local input-to-state stability: Characterizations and counterexamples.* SCL, 2016.
- A.M., F. Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems.* IEEE TAC, 2018.

Theorem (Jacob, Mironchenko, Partington, Wirth, to be submitted, 2019)

Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X.
Assume that there is $P \in L(X)$ satisfying:

(i) *P satisfies $\mathrm{Re}\,\langle Px, x\rangle_X > 0, \quad x \in X\backslash\{0\}$.*

(ii) *P satisfies Lyapunov inequality*

$$\langle (PA + A^* P)x, x\rangle_X \leq -\langle x, x\rangle_X, \quad x \in D(A), \tag{3}$$

(iii) *It holds that $Im(P) \subset D(A^*)$.*

(iv) *PA is bounded.*

Then

$$V(x) := \mathrm{Re}\,\langle Px, x\rangle_X \tag{4}$$

is a non-coercive ISS Lyapunov function for (1) with any ∞-admissible input operator $B \in L(U, X_{-1})$, and thus (1) is ISS for such B.

When does $P := -A^{-1}$ works?

Proposition (Jacob, Mironchenko, Partington, Wirth, to be submitted, 2019)

$$\Sigma : \dot{x} = Ax + Bu.$$

Let:

- *A generate an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on a complex Hilbert space X*
- *$B \in L(U, X_{-1})$ be ∞-admissible.*
- *$D(A) \subseteq D(A^*)$ and*

$$\exists \delta < 1 : \ \forall x \in X \quad \mathrm{Re}\, \langle A^* A^{-1} x, x\rangle_X + \delta \|x\|_X^2 \geq 0. \tag{5}$$

- *$\mathrm{Re}\, \langle Ax, x\rangle_X < 0$ for every $x \in D(A)\backslash\{0\}$.*

Then

$$V(x) := -\mathrm{Re}\, \langle A^{-1} x, x\rangle_X \tag{6}$$

is an ISS Lyapunov function.

Subnormal Operators

Definition

A densely defined operator $(A, D(A)) : X \to X$ on a Hilbert space X is said to be normal if $D(A) = D(A^*)$ and $\|Ax\|_X = \|A^*x\|_X$ for all $x \in D(A)$.

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Definition

A closed, densely-defined operator A on a Hilbert space X is called subnormal, if there is a Hilbert space Z containing X as a subspace and a normal operator $(N, D(N)) : Z \to Z$ so that $A = N|_X$ (the restriction of N to X) and X is an invariant subspace for N, that is, $N(D(N) \cap X) \subseteq X$.

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Example

1. Clearly, every normal operator on a Hilbert space is subnormal.
2. Symmetric operators on Hilbert spaces are subnormal.
3. Isometries are subnormal

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Corollary (Jacob, Mironchenko, Partington, Wirth, CDC, 2018)

Let A generate an exponentially stable analytic semigroup on a Hilbert space X and assume that A is subnormal and $B \in L(\mathbb{C}^n, X_{-1})$ be ∞-admissible. Then

$$V(x) := -\mathrm{Re}\,\langle A^{-1}x, x\rangle_X \tag{7}$$

is a non-coercive ISS Lyapunov function satisfying

$$\dot{V}_u(x) \le -c_1\|x_0\|_X^2 + c_2\|u\|_\infty^2$$

for some constants $c_1, c_2 > 0$ and all $x_0 \in X$ and $u \in L^\infty([0,\infty), U)$.

$$\dot{x} = Ax + Bu$$

Consider this system with:

- $X = l_2(\mathbb{N}) := \left\{x = \{x_k\}_{k=1}^{\infty} : \|x\|_X = \left(\sum_{k=1}^{\infty} x_k^2\right)^{1/2} < \infty\right\}$.
- $Ae_k = -\lambda_k e_k$, where e_k is the k-th unity vector of $l_2(\mathbb{N})$ and $\lambda_k \in \mathbb{R}$.
- Hence A is self-adjoint.
- $\lambda_k < \lambda_{k+1}$ for all k and $\lambda_k \to \infty$ as $k \to \infty$
- A generates an exponentially stable semigroup, i.e. there exists $\varepsilon > 0$: $\lambda_k > \varepsilon$ for all $k > 0$.

Let first B be bounded.

We want to find an ISS Lyapunov function for this system

Bounded B: First ISS LF construction

Above method leads to the following ISS LF:

$$V_1(x) = -\left\langle \frac{1}{2}A^{-1}x, x \right\rangle = \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} \langle x, e_k \rangle^2 . \tag{8}$$

Note: V_1 is not coercive since $\lambda_k \to \infty$ as $k \to \infty$.

Diagonal systems with bounded B

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Bounded B: Second ISS LF construction

For this example it is easy to find another ISS Lyapunov function V_2:

$$V_2(x) := \langle x, x \rangle = \|x\|_X^2, \quad x \in X.$$

Note: V_2 is a coercive ISS LF for any $B \in L(U, X)$.

Can we characterize admissibility via Lyapunov equation?

Now assume that B is merely an ∞-admissible operator.

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$$X_{-1} := \Big\{\{x_k\}_{k=1}^{\infty} : \sum_{k=1}^{\infty} \frac{|x_k|^2}{\lambda_k^2} < \infty\Big\}.$$

Lemma (Jacob, 2018)

B is ∞-admissible for T $\quad\Leftrightarrow\quad$ $\sum_{k=1}^{\infty} \frac{|b_k|^2}{\lambda_k^2} < \infty$.

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Let $P := -A^{-1}$ (a solution of the Lyapunov equation). One can show:

Proposition

$PB \in L(U, X) \quad \Leftrightarrow \quad$ *B is ∞-admissible*

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$PB \in L(U, X) \quad\Leftrightarrow\quad$ *B is ∞-admissible*

Can we similarly characterize admissibility in a non-diagonal case?

ISS LFs for unbounded B

After some further analysis one can show that

$$V_1(x) = -\left\langle \frac{1}{2}A^{-1}x, x \right\rangle = \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} \langle x, e_k \rangle^2 . \tag{9}$$

is a non-coercive ISS LF for any ∞-admissible $B \in L(U, X_{-1})$.

At the same time,

$$V_2(x) := \langle x, x \rangle = \|x\|_X^2, \quad x \in X.$$

is not an ISS Lyapunov function for any unbounded admissible $B \in L(U, X_{-1})$.

Do coercive ISS Lyapunov functions for diagonal systems with unbounded input operators exist?

Example: heat equation with Dirichlet boundary control

Consider the following system ($a > 0$):

$$\begin{aligned} x_t(\xi,t) &= a x_{\xi\xi}(\xi,t), \quad \xi \in (0,1),\ t > 0, \\ x(0,t) &= 0, \quad x(1,t) = u(t), \quad t > 0, \\ x(\xi,0) &= x_0(\xi). \end{aligned}$$

We choose $X = L^2(0,1)$, $U = \mathbb{C}$,

$$Af := f'', \qquad f \in D(A) := \left\{ f \in H^2(0,1) \mid f(0) = f(1) = 0 \right\}, \qquad B := a\delta_1'.$$

No constructions of coercive ISS Lyapunov functions are available for this system.

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- A is a self-adjoint operator on X generating an exponentially stable analytic C_0-semigroup on X
- $B \in X_{-1} = L(U, X_{-1})$ is ∞-admissible
- $\forall x_0 \in X$, $\forall u \in L^\infty(0,\infty)$ the corresponding mild solution is continuous.
- Further, the following ISS-estimates hold $\forall x_0 \in X$, $u \in L^\infty(0,\infty)$:

$$\|x(t)\|_{L^2(0,1)} \leq \mathrm{e}^{-a\pi^2 t}\|x_0\|_{L^2(0,1)} + \frac{1}{\sqrt{3}}\|u\|_{L^\infty(0,t)}$$

The corresponding non-coercive ISS Lyapunov function is:

$$V(x) = -\langle A^{-1}x, x\rangle_X = \int_0^1 \left(\int_\xi^1 (\xi - \tau)x(\tau)d\tau \right) \overline{x(\xi)} d\xi.$$

Overview

We have discussed

- Coercive and non-coercive Lyapunov functions
- CEP $\wedge$ BRS $\wedge$ $\exists$ nc ISS-LF $\quad \Rightarrow \quad$ ISS
- Constructions of nc ISS LF for boundary control systems, in particular:
  - for a heat equation with Dirichlet control.
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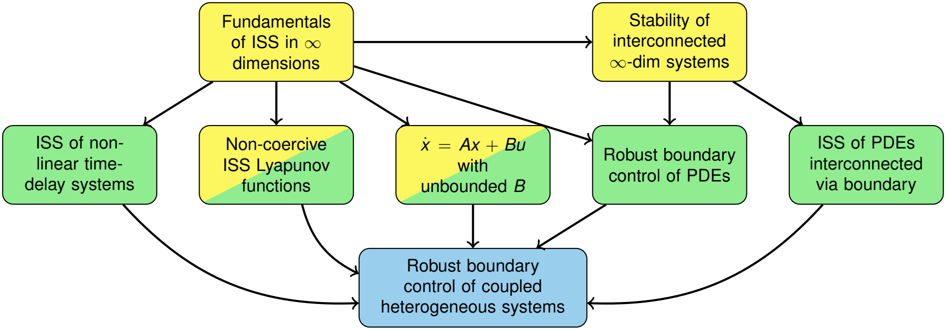
Papers and slides can be found at
www.mironchenko.com

Vision: Robust control of coupled ∞-dim systems



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MCSS Topical Collection: Input-to-state stability for infinite-dimensional systems

Topics of interest

- ISS for partial differential equations
- ISS for boundary control systems
- Lyapunov methods for input-to-state stability
- Applications of ISS to robust control and observation of PDE systems

Important dates

- Deadline for the initial submission of manuscripts: October 1, 2019.
- Notification about the first decision: February 1, 2020.

Guest Editors

- Birgit Jacob, University of Wuppertal
- Andrii Mironchenko, University of Passau
- Felix Schwenninger, University of Hamburg