Stabilization of DAEs via Time-Dependent Switching

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June 25, 2013
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Outline

Notation for DAEs

Consider a regular DAE

$$E\dot{x} = Ax, \tag{1}$$

Theorem (Quasi-Weierstraß form)

If (1) is regular, then there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$:

$$SET = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \quad SAT = \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix},$$

where N, J are square matrices and N is nilpotent.

$$A^d := T\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}T^{-1} \qquad \Pi_{(E,A)} := T\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}T^{-1}.$$

The solution of (1) with $x(0) = x_0 \in C_{(E,A)}$, is also a solution of

$$\dot{x} = A^d x.$$

Switched ODEs

Consider

$$\dot{x} = A_{\sigma(t)} x \,.$$

$$C_{(I,A_i)} \equiv C_{(I,A_j)} = \mathbb{R}^n$$

Solution of switched ODE:

$$x(t) = e^{A_{r_i}(t-t_i)} \cdot e^{A_{r_{i-1}}(t_i-t_{i-1})} \ldots e^{A_{r_1}(t_1-t_0)} x_0,$$

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Let us view (2) as a time-variant DAE

$$E(t)\dot{x} = A(t)x,$$

Solution of (2) (Trenn, 2009):

$$x(t) = e^{A^d_{r_i}(t-t_i)}\Pi_{r_i} \cdot e^{A^d_{r_{i-1}}(t_i-t_{i-1})}\Pi_{r_{i-1}} \cdots e^{A^d_{r_1}(t_1-t_0)}\Pi_{r_1} x_0,$$

Asymptotic stability and stabilization

$$E(t)\dot{x} = A(t)x,$$

Definition (Trenn, 2009)

The zero solution is called globally attractive if $\forall x_0$ solutions of (3), does not contain Dirac impulses and its derivatives and converge to zero when $t \to \infty$.

The zero solution is called globally asymptotically stable (GAS) if it is globally attractive and Lyapunov stable.

Definition

The switched system

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)$$

is called stabilizable via time-dependent switching if there exists a piecewise continuous switching signal $\sigma : \mathbb{R}_+ \to \mathcal{I}$ such that its equilibrium is GAS.

Outline

Approximation of DAEs via ODEs

$$\dot{x} = A_i^\varepsilon x$$

with system matrix

$$A_i^\varepsilon := A_i^d \Pi_i - \frac{1}{\varepsilon}(I - \Pi_i) = T_i \begin{pmatrix} J_i & 0 \\ 0 & -\frac{1}{\varepsilon} I \end{pmatrix} T_i^{-1}.$$

For the set of system matrices A_i^ε we define the switched linear ODE:

$$\dot{x} = A_{\sigma(t)}^\varepsilon x$$

For all $t > 0$ and $x_0 \in \mathbb{R}^n$ it holds

$$(e^{A_i^d t} \Pi_i - e^{A_i^\varepsilon t}) x_0 = e^{-\frac{t}{\varepsilon}} (I - \Pi_i) x_0.$$

Criterion of stabilizability via switching

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)$$

Theorem (Criterion for stabilizability of a switched DAE)

The following statements are equivalent:

1. *sDAE is stabilizable*
2. *sDAE is stabilizable via periodic signal σ_p.*
3. $\exists \varepsilon_0 > 0$*: switched ODE is stabilizable via the (same) periodic switching signal σ_p for all $0 < \varepsilon < \varepsilon_0$, uniformly w.r.t. ε:*

$$\exists s > 0 : \forall \varepsilon \in (0, \varepsilon_0) \Rightarrow |\phi_\varepsilon(s, x_0, \sigma_p)| \leq \frac{1}{2}|x_0|, \forall x_0 \in \mathbb{R}^n.$$

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Lemma

Let A_i^d, $i = 1, \ldots, m$ commute pairwise. If (3) is stabilizable for a certain ε_0, then it is stabilizable via the same switching signal σ_p, uniformly w.r.t. $\varepsilon \in (0, \varepsilon_0)$.

Tightness of the above criterion

Consider the system $\Sigma_1 = (E_1, A_1)$ with

$$E_1 := \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad A_1 := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

and the system $\Sigma_2 = (E_2, A_2)$ with

$$E_2 := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad A_2 := \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Their flow matrices and consistency projectors are as follows:

$$\Pi_1 = A_1^d = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

and

$$\Pi_2 = A_2^d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

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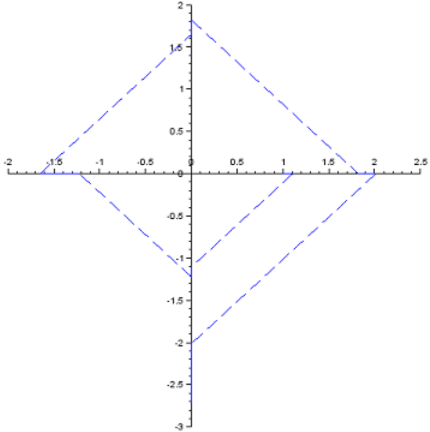

Figure : Typical trajectory of above switched DAE for $x_0 = (1, 0)^T$

Now consider the corresponding ODE approximations

$$A_1^\varepsilon = A_1^d \Pi_1 - \frac{1}{\varepsilon}(I - \Pi_1) = \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -1 - \frac{1}{\varepsilon} & 1 \end{pmatrix}$$

and

$$A_2^\varepsilon = A_2^d \Pi_2 - \frac{1}{\varepsilon}(I - \Pi_2) = \begin{pmatrix} 1 & 1 + \frac{1}{\varepsilon} \\ 0 & -\frac{1}{\varepsilon} \end{pmatrix}.$$

Consider the periodic signal σ_ε of a period 2ε, defined by

$$\sigma_\varepsilon(t) = \begin{cases} 1, & t \in [0, \varepsilon), \\ 2, & t \in [\varepsilon, 2\varepsilon). \end{cases}$$

$$M_\varepsilon := e^{A_2^\varepsilon \varepsilon} e^{A_1^\varepsilon \varepsilon} = \begin{pmatrix} e^{\varepsilon-1} - (e^\varepsilon - e^{-1})^2 & e^\varepsilon(e^\varepsilon - e^{-1}) \\ -e^{-1}(e^\varepsilon - e^{-1}) & e^{\varepsilon-1} \end{pmatrix}.$$

The state of the system (3) at time $2\varepsilon k$, corresponding to σ_ε is given by

$$\phi_{eps}(2\varepsilon k, x_0) = M_\varepsilon^k x_0.$$

Define $y(k) := \phi_{eps}(k2\varepsilon, x_0)$. Then we obtain a discrete system

$$y(k) = M_\varepsilon y(k-1).$$

One can easily compute, that $\rho(M_0) < 1$.

Sufficient condition for stabilizability

Theorem

Assume there exist $\varepsilon_0 > 0$ *and* $s > 0$ *so that for all* $0 < \varepsilon < \varepsilon_0$ *there exists a periodic switching signal* σ_ε *with period* s *such that*

$$|\phi_\varepsilon(s, x_0, \sigma_\varepsilon)| \leq \frac{1}{2}|x_0|.$$

Moreover, assume there exists $t_d > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$*, for any two subsequent switches of* σ_ε *it holds* $|t_i^\varepsilon - t_{i+1}^\varepsilon| \geq t_d$*. Then sDAE is stabilizable via* σ_{ε^*} *for some* $\varepsilon^* \in (0, \varepsilon_0)$*.*

Outline

Fast switching stabilization of ODEs

$$\dot{x} = A_{\sigma(t)} x \,.$$

Theorem

Let $\exists \alpha_j \in [0,1]$, $j = 1, \ldots, m$ *with* $\alpha_1 + \ldots + \alpha_m = 1$*:*

$$S = \alpha_1 A_1 + \ldots + \alpha_m A_m,$$

so that S *is Hurwitz. Then* $\exists h > 0$*:* h*-periodic signal* σ

$$\sigma(t) = k, \quad t \in [\sum_{i=1}^{k-1} \alpha_i, \sum_{i=1}^{k} \alpha_i)$$

stabilizes the switched ODE.

DAE case: role of the fast dynamics

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x\,.$$

Satz (Stabilization via fast switching)

Let $\exists k \geq 1$, $h_0 > 0$, *sequence* $\{i_j\}$, $j = 1, \ldots, k$ *and* $\alpha_j \in (0, 1]$, $j = 1, \ldots, k$ *with* $\alpha_1 + \ldots + \alpha_k = 1$ *so that* $\forall h \in (0, h_0)$

$$\rho(P + Sh) < 1 - ch,$$

for some $c > 0$, *where*

$$P := \Pi_{i_k} \cdots \Pi_{i_1}, \quad S := \sum_{j=k}^{1} \Pi_{i_k} \cdots \Pi_{i_{j+1}} \alpha_j A^d_{i_j} \Pi_{i_{j-1}} \cdots \Pi_{i_1}.$$

Let P *be diagonalizable. Then the switched DAE is stabilizable.*

ODE case

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(t) \in \mathbb{R}^n,$$

Corollary

Let $\exists$ a Hurwitz matrix $S \in conv\{A_1, \ldots, A_m\}$. Then sODE is stabilizable.

Proof.

- $P = I$ (diagonal).
- $S = \alpha_1 A_1 + \ldots + \alpha_m A_m$
- $Spec(I + Sh) = 1 + Spec(S)h$.
- $\rho(I + Sh) = 1 - ch$ for certain $c > 0$ and $h < h_0$, where h_0 is small enough.

□

Stabilization via Projections

Let I be an index set. Denote $S := \{S_i\}_{i \in I}$ and define

$$Pr(S) := \{S_{i_1} \cdots S_{i_r} : \{i_1, \ldots, i_r\} \subset I\}.$$

Define $S_\Pi := \{\Pi_1, \ldots, \Pi_m\}$. We call sDAE stabilizable via projections, if $\exists R \in Pr(S_\Pi)$ with $\rho(R) < 1$.

Proposition

If sDAE is stabilizable via projections, then for all $\tau > 0$ and all $\delta \in (0, 1)$ there exists a τ-periodic stabilizer σ_τ of system sDAE, so that the trajectories of sDAE satisfy

$$|\phi(\tau, x_0, \sigma_\tau)| \leq \delta |x_0|, \quad \forall x_0 \in \mathbb{R}^n.$$

Systems with $\dim C_{(E_i,A_i)} = 1$

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x \,.$$

Proposition

Let $\dim C_{(E_i,A_i)} = 1 \ \forall i = 1,\ldots,m$. Then sDAE is stabilizable iff one of the following conditions hold:

1. Some of the constituent systems is GAS.
2. sDAE is stabilizable via projections.

Conclusions

What we have done

- Criterion of stabilizability of a switched DAE in terms of ODE approximations
- Conditions for stabilizability of switched DAEs via fast switching.
- Criterion for stabilizability of DAEs with commutative vector fields.

Possible directions for future work

- Stabilization of DAEs via state-dependent switching
  1. Lyapunov-type conditions for stabilization
  2. Quadratic stabilization
- Connections between state-dependent and time-dependent stabilization.