Input-to-state stability of distributed parameter systems

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Outline

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \ x(t) \in D(A) \subset X, u(t) \in U, \\ x(0) = \phi_0. \end{cases}$$

- X = State space
- $U_c = PC(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.
- $f(0,0) = 0$.

$x \in C([0,T], X)$ is a mild solution iff

$$x(t) = T(t)\phi_0 + \int_0^t T(t-s)f(x(s), u(s))ds.$$

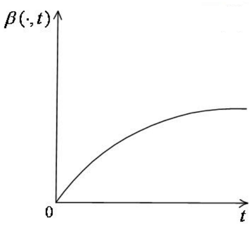
Examples of generators

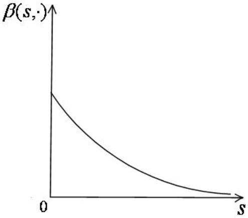
$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \; x(t) \in D(A) \subset X, u(t) \in U, \\ x(0) = \phi_0. \end{cases}$$

- ODEs: A is a matrix, $T(t) = e^{tA}$.
- Parabolic equations: $A = \Delta$
- Hyperbolic equations: $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$
- Schrödinger equation: $A = i\Delta$
- Delay equations: A is a delay operator.

Comparison functions

$$\mathcal{K}_\infty := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma(0) = 0,\ \gamma \text{ is continuous, growing and unbounded}\}$$
$$\mathcal{L} := \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly decreasing and } \lim_{t\to\infty} \gamma(t) = 0\right\}$$
$$\mathcal{KL} := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \ \beta(\cdot, t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r, \cdot) \in \mathcal{L},\ \forall r > 0\}$$





Definition (GAS uniform w.r.t. state (0-UGASx))

0-UGASx $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}$: $\forall \phi_0 \in X, \forall t \geq 0$

$$\|\phi(t, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t).$$

Input-to-state stability

Definition (GAS uniform w.r.t. state (0-UGASx))

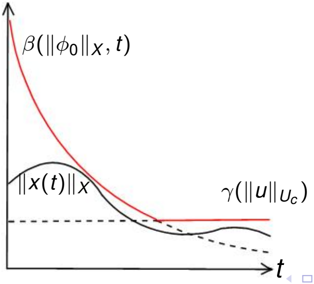
0-UGASx $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}: \quad \forall \phi_0 \in X, \forall t \geq 0$

$$\|\phi(t, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t).$$

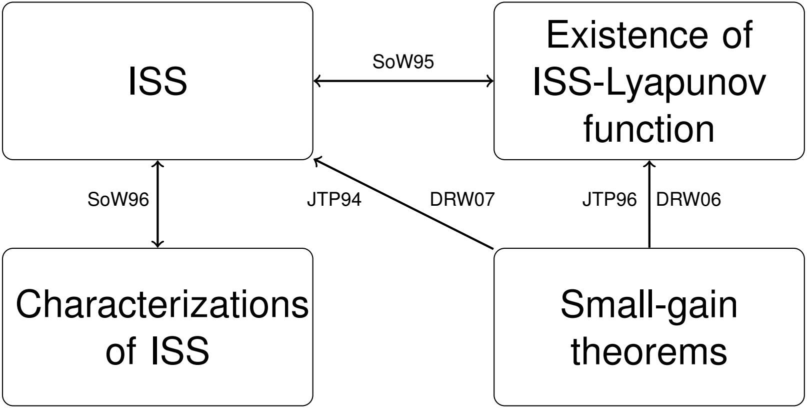
Definition (ISS)

ISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall \phi_0 \in X, \forall u \in U_c$

$$\|\phi(t, \phi_0, u)\|_X \leq \max\{\beta(\|\phi_0\|_X, t), \underbrace{\gamma}_{Gain}(\|u\|_{U_c})\}.$$



Fundamentals of ISS-Theory for ODEs



Stability concepts for ∞-dim systems

$$\Sigma : \quad \dot{x} = Ax + Bu, \quad x(0) = x_0, B \in L(U, X).$$

$$\phi(t, x_0, u) = T(t)x_0 + \int_0^t T(t-r)Bu(r)dr,$$

Fact

0-UGASx $\quad \Leftrightarrow \quad \exists M, \lambda > 0 : \|T(t)\| \leq Me^{-\lambda t} \Leftrightarrow$ T exp. stable .

Fact

0-GAS $\quad \Leftrightarrow \quad \lim_{t\to\infty} \|T(t)x\| = 0 \ \forall x \in X \Leftrightarrow$ T strongly stable.

Stability concepts for ∞-dim systems

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For ∞-dim systems: GAS $\neq$ 0-UGASx.

ISS of linear systems for $U_c := PC(\mathbb{R}_+, U)$

$$\Sigma: \quad \dot{x} = Ax + Bu, \quad x(0) = x_0.$$

$$\phi(t, x_0, u) = T(t)x_0 + \int_0^t T(t-r)Bu(r)dr,$$

Fact

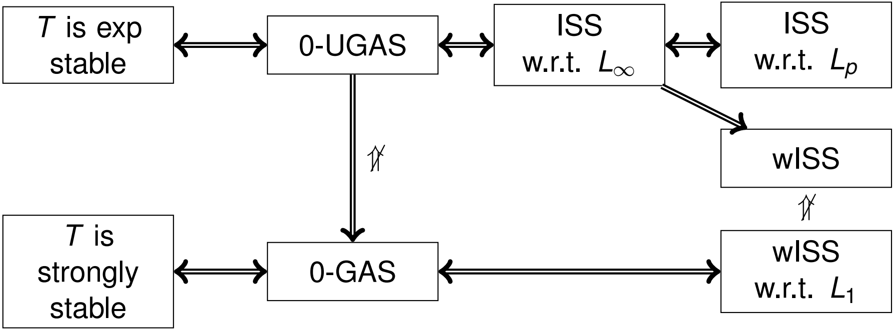
$$\Sigma \text{ is 0-UGAS}x \quad \Leftrightarrow \quad \exists M, \lambda > 0 : \|T(t)\| \leq Me^{-\lambda t}.$$

$$\begin{aligned} \|\phi(t, x_0, u)\|_X &\leq \|T(t)\|\|x_0\|_X + \int_0^t \|T(t-r)\|\|B\|\|u(r)\|_U dr, \\ &\leq \underbrace{Me^{-\lambda t}\|x_0\|_X}_{\beta(\|x_0\|_X, t)} + K \sup_{r\in[0,t]} \|u(r)\|_U, \quad K > 0. \end{aligned}$$

Lemma

Σ is 0-UGASx $\Leftrightarrow$ *Σ is ISS w.r.t. L_∞* $\overset{\Rightarrow}{\not\Leftarrow}$ *Σ is 0-GAS*

ISS theory for linear systems with bounded input operators



Lyapunov functions

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \ x(t) \in D(A) \subset X, u(t) \in U, \\ x(0) = \phi_0. \end{cases}$$

Definition

$V : X \to \mathbb{R}_+$ is ISS-Lyapunov function iff $\exists \psi_1, \psi_2, \chi, \alpha \in \mathcal{K}_\infty$:

- $\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X)$
- $V(x) \geq \chi(\|\xi\|_U) \quad \Rightarrow \quad \dot{V}_u(x) \leq -\alpha(V(x))$,

$\forall x \in X, \ \forall \xi \in U, \ \forall u \in U_c$ with $u(0) = \xi$.
Here $\dot{V}_u(x) = \overline{\lim\limits_{t \to +0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x))$.

Theorem

$\exists$ *ISS-Lyapunov function* $\Rightarrow$ *ISS.*

Example

$$\begin{cases} \frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} - f(s) + u(x,t), \quad x \in (0,\pi),\ t > 0, \\ s(0,t) = s(\pi,t) = 0. \end{cases}$$

Here f is locally Lipschitz, uneven and monotonically increasing up to infinity.

Let $u(\cdot,t) \in L_2(0,\pi)$. Define

$As = \frac{d^2 s}{dx^2}$ with $D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$.

$$\frac{ds}{dt} = As - f(s) + u,\ t > 0.$$

This equation defines a control system with $X = H_0^1(0,\pi)$ and $U = L_2(0,\pi)$.

$$V(s)=\int_0^\pi\left(\frac{1}{2}s_x^2(x)+\int_0^{s(x)}f(y)dy\right)dx.$$

First property of LF:

$$\int_0^{s(x)}f(y)dy\geq 0\ \Rightarrow\ V(s)\geq\int_0^\pi\frac{1}{2}s_x^2(x)dx=\frac{1}{2}\|s\|^2_{H_0^1(0,\pi)}.$$

The derivative of V along the trajectories is equal

$$\dot V(s)=\underbrace{-\int_0^\pi(s_{xx}(x)-f(s(x)))^2dx}_{I(s)}+\int_0^\pi(s_{xx}(x)-f(s(x)))(-u)dx.$$

Using Cauchy-Schwarz inequality for the second term, we have:

$$\dot V(s)\leq -I(s)+\sqrt{I(s)}\,\|u\|_{L_2(0,\pi)}.$$

For $s \in H^1_0(0,\pi) \cap H^2(0,\pi)$ using Friedrich's inequality one can prove:

$$I(s) \geq \int_0^\pi s_{xx}^2(x)dx \geq \int_0^\pi s_x^2(x)dx = \|s\|^2_{H^1_0(0,\pi)}.$$

Define the gain as $\chi(r) = ar,\ a > 1$.
Assuming

$$\|s\|_{H^1_0(0,\pi)} \geq \chi(\|u\|_{L_2(0,\pi)})$$

we obtain

$$\dot{V}(s) \leq -I(s) + \frac{1}{a}\sqrt{I(s)}\|s\|_{H^1_0(0,\pi)} \leq -(1-\frac{1}{a})I(s) \leq -(1-\frac{1}{a})\|s\|^2_{H^1_0(0,\pi)}.$$

This proves, that V is an ISS-Lyapunov function.

Interconnected systems

$$\Sigma : \begin{cases} \Sigma_i : \quad \dot{x}_i = A_i x_i + f_i(x_1, \ldots, x_n, u), \;\; x_i \in X_i \\ i = 1, \ldots, n \end{cases}$$

- X_i state space of Σ_i
- A_i infinitesimal generator of C_0-semigroup on X_i.
- $X = X_1 \times \ldots \times X_n$ state space of the whole system.
- $\tilde{X}_i := X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n \times U$ space of inputs into i-th subsystem.

ISS-Lyapunov functions for subsystems

$$\Sigma : \begin{cases} \Sigma_i : \quad \dot{x}_i = A_i x_i + f_i(x_1, \ldots, x_n, u), \;\; x_i \in X_i \\ i = 1, \ldots, n \end{cases}$$

ISS-LF for Σ_i

$V_i : X_i \to \mathbb{R}_+$ is ISS-Lyapunov function for Σ_i iff
$\exists \psi_{i1}, \; \psi_{i2}, \; \alpha_i, \; \chi_i, \; \chi_{ij} \in \mathcal{K}_\infty, \quad j = 1, \ldots, n$:

- $\psi_{i1}(\|x_i\|_{X_i}) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|_{X_i})$
- $V_i(x_i) \geq \max\{ \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U)\} \;\Rightarrow\; \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)),$

$\forall x_i \in X_i, \; \forall \tilde{x}_i \in \tilde{X}_i, \forall v \in PC(\mathbb{R}_+, \tilde{X}_i)$ with $v(0) = \tilde{x}_i$.

Small-gain theorem

Gain matrix: $\Gamma_M = (\chi_{ij})_{i,j=1,\ldots,n}$, $\chi_{ij} \in \mathcal{K}_\infty \cup \{0\}$.
Gain operator: $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$

$$\Gamma(s) := \left(\max_{j=1}^{n} \chi_{1j}(s_j), \ldots, \max_{j=1}^{n} \chi_{nj}(s_j) \right), \; s \in \mathbb{R}^n_+.$$

Theorem (Dashkovskiy, M., MCSS, 2013)

Let V_i be ISS-Lyapunov function for Σ_i with gains χ_{ij}.

$$\Gamma(s) \not\geq s, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\} \qquad \text{(SGB)}$$

$\Rightarrow$

- *Σ is ISS.*
- *$V(x) := \max_i\{\sigma_i^{-1}(V_i(x_i))\}$ is a Lyapunov function for Σ.*

For $n = 2$ (SGB) $\quad \Leftrightarrow \quad \chi_{12} \circ \chi_{21} < id$.

Example: interconnected diffusion equations

$$\begin{cases} \frac{\partial s_1}{\partial t} = c_1 \frac{\partial^2 s_1}{\partial x^2} + a_{12} s_2, \quad x \in (0,d),\ t > 0, \\ s_1(0,t) = s_1(d,t) = 0; \\ \frac{\partial s_2}{\partial t} = c_2 \frac{\partial^2 s_2}{\partial x^2} + a_{21} s_1, \quad x \in (0,d),\ t > 0, \\ s_2(0,t) = s_2(d,t) = 0. \end{cases}$$

State spaces:

$$X_1 = X_2 = L_2(0,d)$$

Generators:

$$A_i = c_i \frac{d^2}{dx^2} \quad \text{with} \quad D(A_i) = H_0^1(0,d) \cap H^2(0,d).$$

ISS-Lyapunov functions for subsystems:

$$V_i(s_i) = \frac{1}{2c_i} \left(\frac{d}{\pi} \right)^2 \|s_i\|^2_{L_2(0,d)}.$$

$$\begin{aligned}
\dot{V}_1(s_1) &= \frac{1}{c_1}\left(\frac{d}{\pi}\right)^2 \int_0^d s_1(x)\left(c_1\frac{\partial^2 s_1}{\partial x^2} + a_{12}s_2(x)\right)dx \\
&= -\left(\frac{d}{\pi}\right)^2 \int_0^d \left(\frac{\partial s}{\partial x}\right)^2 dx + \frac{1}{c_1}\left(\frac{d}{\pi}\right)^2 \int_0^d a_{12}s_1(x)s_2(x)dx \\
&\leq -\|s_1\|^2_{L_2(0,d)} + \frac{1}{c_1}\left(\frac{d}{\pi}\right)^2 |a_{12}|\|s_1\|_{L_2(0,d)}\|s_2\|_{L_2(0,d)}.
\end{aligned}$$

Analogously we obtain:

$$\dot{V}_2(s_2) \leq -\|s_2\|^2_{L_2(0,d)} + \frac{1}{c_2}\left(\frac{d}{\pi}\right)^2 |a_{21}|\|s_1\|_{L_2(0,d)}\|s_2\|_{L_2(0,d)}.$$

Gains:

$$\chi_{12}(r) = \frac{c_2}{c_1^3}\left(\frac{d}{\pi}\right)^4 \left|\frac{a_{12}}{1-\varepsilon}\right|^2 \cdot r, \quad \chi_{21}(r) = \frac{c_1}{c_2^3}\left(\frac{d}{\pi}\right)^4 \left|\frac{a_{21}}{1-\varepsilon}\right|^2 \cdot r.$$

Thus:

$$V_1(s_1) \geq \chi_{12} \circ V_2(s_2) \quad \Rightarrow \quad \dot{V}_1(s_1) \leq -\varepsilon \|s_1\|^2_{L_2(0,d)},$$

$$V_2(s_2) \geq \chi_{21} \circ V_1(s_1) \quad \Rightarrow \quad \dot{V}_2(s_2) \leq -\varepsilon \|s_2\|^2_{L_2(0,d)}.$$

Small-gain condition:

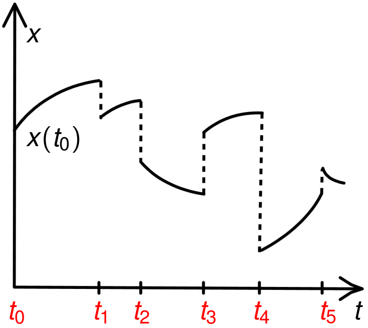
$$\chi_{12} \circ \chi_{21} < \mathrm{Id} \quad \Leftrightarrow \quad \frac{1}{c_1^2 c_2^2} \left(\frac{d}{\pi}\right)^8 \frac{|a_{12}a_{21}|^2}{(1-\varepsilon)^4} < 1,$$

Thus, if

$$|a_{12}a_{21}| < c_1 c_2 \left(\frac{\pi}{d}\right)^4$$

$\Rightarrow$ 0-UGASx.

Impulsive systems



$$\dot{x}(t) = Ax(t) + f(x(t), u(t)) \quad , \; t \notin \{t_1, t_2, \ldots\},$$
$$x(t) = g(x^-(t), u^-(t)) \quad , \; t \in \{t_1, t_2, \ldots\}.$$

$u \in PC([0,\infty), U)$, $\quad x(t) \in X$, $\quad f : X \times U \to X$. $x^-(t) := \lim_{s \nearrow t} x(s)$,

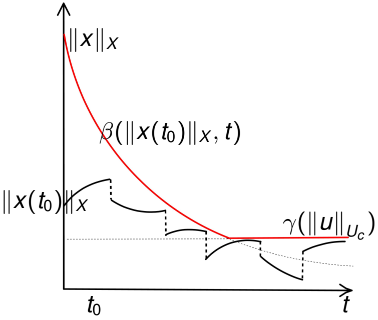
$u^-(t) := \lim_{s \nearrow t} u(s)$.

Definition (Input-to-state stability (ISS))

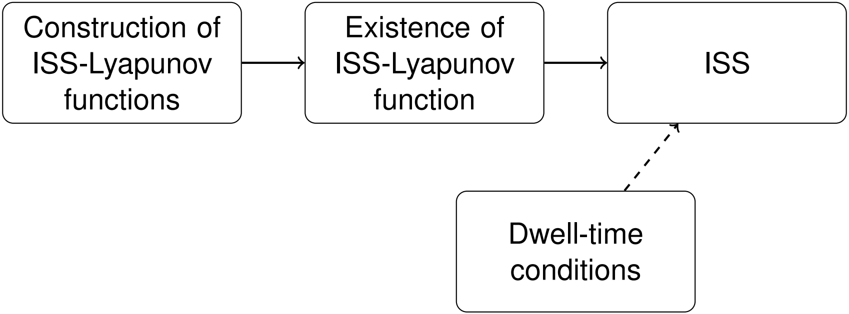
ISS (for given T) $\quad :\Leftrightarrow \quad \exists \beta \in \mathcal{KL},\ \gamma \in \mathcal{K}_\infty$: $\forall x \in X, \forall u \in U_c, \forall t \geq t_0$

$$\|\phi(t, t_0, x, u)\|_X \leq \max\{\beta(\|x\|_X, t - t_0), \gamma(\|u\|_{U_c})\}.$$

ISS uniform w.r.t. $S \quad :\Leftrightarrow \quad$ ISS $\forall T \in S$, and β, γ do not depend on $T \in S$.



ISS-Theory for ∞-dim systems



ISS-Lyapunov functions (ISS-LF)

$$\Sigma: \quad \begin{array}{l} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \; t \neq t_k, \\ x(t) = g(x^-(t), u^-(t)), \; t = t_k, \; k \in \mathbb{N}. \end{array}$$

Definition (ISS-Lyapunov function for impulsive systems)

$V : X \to \mathbb{R}_+$ is ISS-Lyapunov function for Σ if $\exists\, \psi_1, \psi_2 \in \mathcal{K}_\infty$:

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad x \in X$$

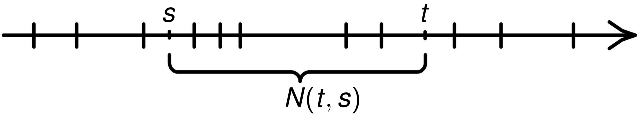
and $\exists \chi, \alpha, \varphi \in \mathcal{K}_\infty$: $\forall x \in X$, $\forall \xi \in U$ and $\forall u \in U_c$ with $u(0) = \xi$

$$V(x) \geq \chi(\|\xi\|_U) \quad \Rightarrow \quad \begin{cases} \dot{V}_u(x) \leq -\varphi(V(x)) \\ V(g(x,\xi)) \leq \alpha(V(x)), \end{cases}$$

V is exponential ISS-LF for Σ with coefficients $c, d \in \mathbb{R}$, if

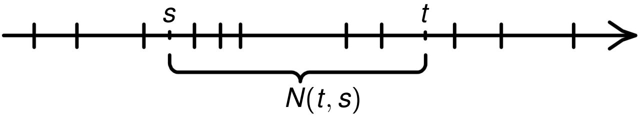
$$V(x) \geq \chi(\|\xi\|_U) \quad \Rightarrow \quad \begin{cases} \dot{V}_u(x) \leq -cV(x) \\ V(g(x,\xi)) \leq e^{-d} V(x). \end{cases}$$

Average dwell time (ADT) condition



Let $N(t,s)$ be number of impulse times t_k in $(s,t]$.

Average dwell time (ADT) condition



Let $N(t,s)$ be number of impulse times t_k in $(s,t]$.

Theorem (Hespanha, Liberzon, Teel, Automatica 2008)

Let V be exponential ISS-LF for Σ with coefficients $c, d \in \mathbb{R}$, $d \neq 0$.
$\forall\ \mu, \lambda > 0$ $\quad \mathcal{S}[\mu,\lambda]$: $\quad \Leftrightarrow \quad$ *class of impulse time sequences $\{t_k\}$:*

$$-dN(t,s) - (c-\lambda)(t-s) \leq \mu \qquad \forall s,t : 0 \leq s \leq t. \qquad \text{(ADT)}$$

Then Σ is uniformly ISS w.r.t. $\mathcal{S}[\mu,\lambda]$.

Generalized ADT condition

Theorem (Dashkovskiy, M., SICON, 2013)

Let V be exponential ISS-LF for Σ with coefficients $c, d \in \mathbb{R}$, $d \neq 0$.
$\forall\ h : \mathbb{R}_+ \to (0,\infty)$: $\exists g \in \mathcal{L}$: $h(x) \leq g(x)\ \forall x \in \mathbb{R}_+$
$\mathcal{S}[h]$:⇔ *class of impulse time sequences:*

$$-dN(t,s) - c(t-s) \leq \ln h(t-s) \quad \forall t \geq s \geq t_0. \qquad \text{(gADT)}$$

Then Σ is uniformly ISS w.r.t. $\mathcal{S}[h]$.

Corollary

gADT with $h(x) = e^{\mu - \lambda x}$, $x \in \mathbb{R}_+ \quad \Rightarrow \quad$ ADT.

ISS and non-exponential ISS-Lyapunov functions

$$V(x) \geq \gamma(\|u\|_U) \Rightarrow \begin{cases} \dot{V}_u(x) \leq -\varphi(V(x)) \\ V(g(x,u)) \leq \alpha(V(x)). \end{cases}$$

Let $S_\theta := \Big\{\{t_i\}_1^\infty \subset [t_0, \infty) \ : \ t_{i+1} - t_i \geq \theta, \ \forall i \in \mathbb{N}\Big\}$.

Theorem (Dashkovskiy, M., SICON, 2013)

Let V be an ISS-Lyapunov function for Σ.

$$\exists \theta, \delta > 0 : \quad \int_r^{\alpha(r)} \frac{ds}{\varphi(s)} \leq \theta - \delta, \quad \forall r > 0 \qquad \text{(FDT)}$$

$\Rightarrow$ *ISS $\forall$ sequences of impulse times $T \in S_\theta$.*

Overview of dwell-time conditions

for exponential LFs

for nonexponential LFs

generalized ADT

$$-dN(t,s) - c(t-s) \le \ln h(t-s)$$

$$h(x) \;:=\; e^{\mu-\lambda x}$$

Average DT

$$-dN(t,s) - (c-\lambda)(t-s) \le \mu$$

$$\mu \;:=\; -d$$

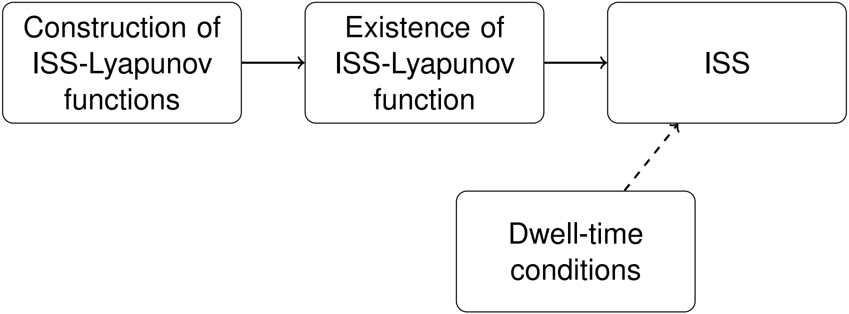
Fixed DT

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \;\le\; \theta - \delta$$

$$\varphi \;:=\; c \cdot id$$
$$\alpha \;:=\; e^{-d} \cdot id$$

$$\frac{1}{\theta} \;\le\; \frac{c-\lambda}{-d}$$

Summary and Outlook



Other results in ISS theory

- Linearization method for study of LISS
- Small gain theorems for impulsive systems
- Small gain theorems for time-delay systems

Plans for the near future

- ISS of linear systems $\dot{x}(t) = Ax(t) + Cu(t)$, with unbounded C.
- Converse ISS Lyapunov theorem
- Characterisation of ISS for ∞-dim systems.
- Applications of ISS Theory
- Robust Stabilisation of PDEs
- "Integral ISS" theory for ∞-dim systems.