Local ISS of Reaction-Diffusion Systems

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Main results

- Generalisation of Lyapunov methods and Linearisation technique to infinite-dimensional ISS systems
- Small-gain theorem for interconnections of infinite-dimensional ISS systems.

Control-theoretic Framework

Let $(X, \|\cdot\|_X)$ be a state space, $(U, \|\cdot\|_U)$ be an input space and U_c be the set of admissible input functions: $\mathbb{R}_+ \to U$.

Definition 1. *The triple $\Sigma = (X, U_c, \phi)$ is a control system, if:*

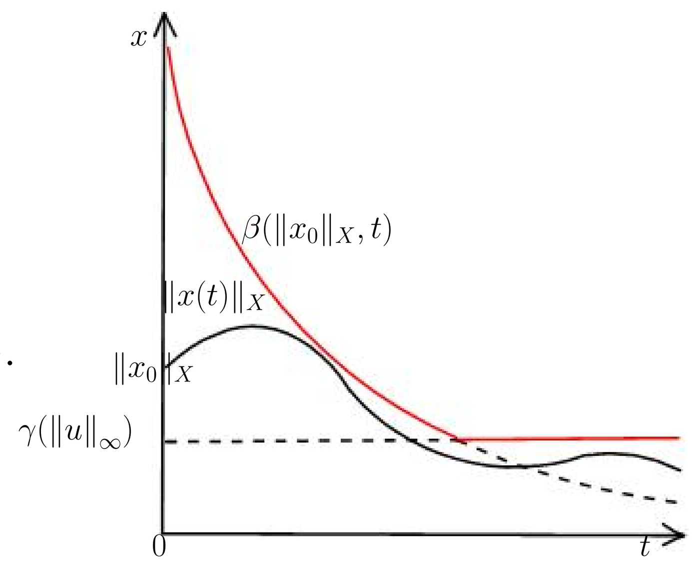
- $\phi(t,t,x,\cdot) = x$ *for all* $t \geq 0$.
- $\forall\, t \geq r \geq s \geq 0,\ \forall x \in X,\ \forall u_1 \in U_c^{[s,r]},\ u_2 \in U_c^{[r,t]}$ *it holds* $\phi(t,r,\phi(r,s,x,u_1),u_2) = \phi(t,s,x,u)$, *where* $u(\tau) := \begin{cases} u_1(\tau), \tau \in [s,r], \\ u_2(\tau), \tau \in [r,t]. \end{cases}$
- $\forall x \in X,\ u \in U_c$ *the map* $t \to \phi(t,0,x,u)$ *is in* $C([0,\infty), X)$
- ϕ *is continuous w.r.t. two last arguments.*

Let $\Sigma = (X, U_c, \phi)$ be time-invariant and $\phi(t,0,0,0) \equiv 0$.

Definition 2. *Σ is locally input-to-state stable (LISS), if $\exists \rho_x, \rho_u > 0$ and $\exists \beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$, such that $\forall t \geq 0$, $\forall \phi_0 : \|\phi_0\|_X \leq \rho_x$ and $\forall u \in U_c$: $\|u\|_{U_c} \leq \rho_u$ it holds*

$$\|\phi(t,0,\phi_0,u)\|_X \leq \max\{\beta(\|\phi_0\|_X,t), \gamma(\|u\|_{U_c})\}.$$

If $\beta(r,t) = Me^{\omega t}r$, for some $\omega < 0$, then (X, U_c, ϕ) is locally exponentially ISS



Lyapunov functions

Definition 3. *A smooth function $V : D \to \mathbb{R}_+$, $D \subset X$, $0 \in int(D)$ is called LISS-LF for system (X, U_c, ϕ), if there exist $\rho_x, \rho_u > 0$, $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and a positive definite function α, such that:*

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in D$$

and $\forall x \in D : \|x\|_X \leq \rho_x$, $\forall u \in U : \|u\|_U \leq \rho_u$ it holds:

$$\|x\|_X \geq \chi(\|u\|_U) \ \Rightarrow\ \dot{V}(x) \leq -\alpha(\|x\|_X),$$

where

$$\dot{V}(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t,0,x,u)) - V(x)).$$

Function χ is called Lyapunov gain.

Theorem 1. *Let $\Sigma = (X, U_c, \phi)$ be a time-invariant control system. If Σ possesses a LISS-Lyapunov function, then Σ is LISS.*

Linearisation Method

Let X be a Hilbert space, and let A generate an analytic semigroup on X. Consider a system

$$\dot{x} = Ax + f(x,u),\ x(t) \in X, u(t) \in U. \tag{1}$$

Theorem 2. *Let for some $B \in L(X)$ and $C \in L(U,X)$ it holds*

$$f(x,u) = Bx + Cu + g(x,u).$$

Let $\forall w > 0\ \exists \rho > 0$, s.t. $\forall x, u : \|x\|_X \leq \rho,\ \|u\|_U \leq \rho$ it holds

$$\|g(x,u)\|_X \leq w(\|x\|_X + \|u\|_U).$$

If the system

$$\dot{x} = Ax + Bx + Cu$$

is exponentially ISS, then (1) is LISS.

Interconnections of infinite-dimensional systems

Let X_i, $i = 1, \ldots, n$ be Banach and A_i generate C_0-semigroup on X_i.

$$\Sigma : \begin{cases} \Sigma_i : \quad \dot{x}_i = A_i x_i + f_i(x_1, \ldots, x_n, u),\ x_i \in X_i \\ i = 1, \ldots, n \end{cases} \tag{2}$$

A space $X = X_1 \times \ldots \times X_n$ is Banach with $\|\cdot\|_X := \|\cdot\|_{X_1} + \ldots + \|\cdot\|_{X_n}$.

$$\Sigma : \quad \dot{x} = Ax + f(x,u)$$

A smooth function $V_i : X_i \to \mathbb{R}_+$, is an ISS-Lyapunov function (ISS-LF) for i-th subsystem of (2), if $\exists \psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, $\chi_{ij}, \chi_i \in \mathcal{K}$, $j = 1, \ldots, n$, and positive definite functions α_i, such that $\forall x_i \in X_i$ holds

$$\psi_{i1}(\|x_i\|_{X_i}) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|_{X_i}),$$

$$V_i(x_i) \geq \max\{\max_j \chi_{ij}(V_j(x_j)), \chi_i(\|u\|_U)\} \ \Rightarrow\ \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)),$$

$$\text{where } \dot{V}(x_i) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi_i(t,0,x_i,u))) - V(x_i)).$$

Let $\Gamma_M = (\chi_{ij})_{i,j=1,\ldots,n}$, $\chi_{ij} \in \mathcal{K}_\infty \cup \{0\}$ (gain matrix).

Let us introduce the gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(s) := \left(\max_j \chi_{1j}(s_j), \ldots, \max_j \chi_{nj}(s_j)\right),\ s \in \mathbb{R}^n_+. \tag{3}$$

Theorem 3. *Let for all Σ_i there exist ISS-Lyapunov function V_i with corresponding gains χ_{ij}. If $\Gamma(s) \not\geq s,\ \forall\, s \in \mathbb{R}^n_+ \setminus \{0\}$, then Σ is ISS and possesses ISS-Lyapunov function, defined by*

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}.$$

Example

$$\begin{cases} \frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} - f(s) + u(x,t), \quad x \in (0,\pi),\ t > 0, \\ s(0,t) = s(\pi,t) = 0. \end{cases} \tag{4}$$

Here f is locally Lipschitz, uneven and monotonically increasing up to infinity.

Let $u(\cdot,t) \in L_2(0,\pi)$. Define $As = \frac{d^2 s}{dx^2}$ with $D(A) = H^1_0(0,\pi) \cap H^2(0,\pi)$.

$$\frac{ds}{dt} = As - f(s) + u,\ t > 0. \tag{5}$$

Equation (5) defines a control system with $X = H^1_0(0,\pi)$ and $U = L_2(0,\pi)$.

$$V(s) = \int_0^\pi \left(\frac{1}{2}s_x^2(x) + \int_0^{s(x)} f(y)dy\right) dx. \tag{6}$$

First property of LF: $\int_0^{s(x)} f(y)dy \geq 0 \ \Rightarrow\ V(s) \geq \int_0^\pi \frac{1}{2}s_x^2(x)dx = \frac{1}{2}\|s\|^2_{H^1_0(0,\pi)}$.

The derivative of V along the trajectories is equal

$$\dot{V}(s) = -\underbrace{\int_0^\pi (s_{xx}(x) - f(s(x)))^2 dx}_{I(s)} + \int_0^\pi (s_{xx}(x) - f(s(x)))(-u)dx.$$

Using Cauchy-Schwarz inequality for the second term, we have:

$$\dot{V}(s) \leq -I(s) + \sqrt{I(s)}\ \|u\|_{L_2(0,\pi)}. \tag{7}$$

For $s \in H^1_0(0,\pi) \cap H^2(0,\pi)$ using Friedrich's inequality one can prove:

$$I(s) \geq \int_0^\pi s_{xx}^2(x)dx \geq \int_0^\pi s_x^2(x)dx = \|s\|^2_{H^1_0(0,\pi)}.$$

Define the gain as $\chi(r) = ar$, $a > 1$. Assuming $\|s\|_{H^1_0(0,\pi)} \geq \chi(\|u\|_{L_2(0,\pi)})$ we obtain

$$\dot{V}(s) \leq -I(s) + \frac{1}{a}\sqrt{I(s)}\|s\|_{H^1_0(0,\pi)} \leq -(1-\frac{1}{a})I(s) \leq -(1-\frac{1}{a})\|s\|^2_{H^1_0(0,\pi)}.$$

This proves, that V is an ISS-Lyapunov function.

Literature

- Curtain, R.F. and Zwart, H. (1995). *An introduction to infinite-dimensional linear systems theory*, Springer-Verlag, New York.
- Dashkovskiy, S.N., Rüffer, B.S., and Wirth, F.R. (2010). Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions. *SIAM Journal on Control and Optimization*, 48(6), 4089–4118.

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