An introduction to input-to-state stability theory

Lecture Notes

Andrii Mironchenko

December 10, 2015

Contents

List of mathematical symbols

$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integer numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
S^n	$\underbrace{S \times \ldots \times S}_{n \text{ times}}$
x^T	transposition of a vector $x \in \mathbb{R}^n$
$\lvert \cdot \rvert$	the norm in the space $\mathbb{R}^s$, $s \in \mathbb{N}$
∇f	gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$.
$f \circ g$	composition of maps f and g
∂G	boundary of a domain G
$L(X,U)$	space of bounded linear operators from X to U
$L(X)$	$= L(X,X)$
$C(X,U)$	space of continuous functions from X to U with finite norm $\lVert u \rVert_{C(X,U)} := \sup\limits_{x \in X} \lVert u(x) \rVert_U$
$PC(\mathbb{R}_+,U)$	space of piecewise continuous (right-continuous) functions from $\mathbb{R}_+$ to U with finite norm $\lVert u \rVert_{PC(\mathbb{R}_+,U)} = \lVert u \rVert_{C(\mathbb{R}_+,U)}$
$AC(\mathbb{R}_+,U)$	space of absolutely continuous functions from $\mathbb{R}_+$ to U with a finite norm $\lVert u \rVert_{C(X,U)}$
$C(X)$	$= C(X,X)$
$C_0(\mathbb{R})$	$\{f \in C(\mathbb{R}) : \forall \varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \mathbb{R} : \lvert f(s) \rvert < \varepsilon \ \forall s \in \mathbb{R} \backslash K_\varepsilon\}$
μ	Lebesgue measure on $\mathbb{R}$.
$L_\infty(\mathbb{R}_+,\mathbb{R}^m)$	the set of Lebesgue measurable functions with finite norm $\lVert f \rVert_\infty := \operatorname{ess\,sup}_{x \geq 0} \lvert f(x) \rvert = \inf_{D \subset \mathbb{R}_+,\ \mu(D)=0} \sup_{x \in \mathbb{R}_+ \backslash D} \lvert f(x) \rvert$
$C_0^k(0,d)$	space of k times continuously differentiable functions $f : (0,d) \to \mathbb{R}$ with a support, compact in $(0,d)$.
$L_p(0,d)$	space of p-th power integrable functions $f : (0,d) \to \mathbb{R}$ with the norm $\lVert f \rVert_{L_p(0,d)} = \left(\int_0^d \lvert f(x) \rvert^p dx \right)^{\frac{1}{p}}$

Preface

Input-to-state stability unified the Lyapunov and input-output stability theories, and evolved to the dominated stability concept in nonlinear control theory with applications to robotics, mechatronics, aerospace engineering, systems' biology to name a few. ISS plays an important role in constructive nonlinear control [49]; in particular, in robust stabilization of nonlinear systems [24], stabilization via controllers with saturation [76], design of robust (in terms of errors in measurements and/or quantization) nonlinear observers [54], nonlinear detectability [72, 51], ISS feedback redesign [68], stability of nonlinear networked control systems [38], [22], supervisory adaptive control [29] and others.

Although ISS theory of systems of ordinary differential equations has reached its maturity, there is still no introductory course in ISS theory. These lecture notes should close this gap and provide a self-contained introduction to the input-to-state stability (ISS) theory, including the foundational results in ISS theory of systems of ordinary differential equations.

In Section 2 we develop a stability theory for classical dynamical systems. The choice of topics is highly selective and its primary objective is to ensure a firm basis for the subsequent development of the input-to-state stability theory. We present a characterization of the global asymptotic stability in terms of comparison functions and in terms of uniform attractivity times and give a simple proof of a converse non-smooth Lyapunov theorem based on a Sontag's lemma on $\mathcal{KL}$-functions. In the end of the section we discuss limitations of the classical dynamical systems theory and argue that ISS can overcome these difficulties.

In Section 3 we develop a Lyapunov theory for the ISS property, including Lyapunov characterization of ISS and local ISS (which generalizes a well-known Massera's theorem for the systems without disturbances) and linearization method. Next we prove several equivalent characterizations of ISS property in terms of uniform attraction times, robust stability as well as an ISS superposition theorem, which states that ISS is equal to a combination of global asymptotic stability and so-called asymptotic gain property. Next in Section 4 we consider large-scale interconnections of control systems. Our main question is under which conditions the system, consisting of ISS components, is itself ISS. We show that a so-called small-gain condition ensures that such a network is ISS. Moreover, the small-gain methods help to construct an ISS Lyapunov function for the whole interconnection, if we know ISS Lyapunov functions for its subsystems.

In Conclusion we give references to other literature which the reader may consult to deepen the knowledge in particular subjects of ISS theory. In addition we give a brief overview of the ISS theory for other classes of systems, as systems of partial differential equations, hybrid, impulsive and time-delay systems.

This script has been prepared for a short course on input-to-state stability held at Odessa I.I. Mechnikov National University (Odessa, Ukraine) during September and October 2015.

This script is a 'work in progress', and thus contains mistakes, typos, inconsistencies etc. and will be improved/expanded/corrected during next months.

The newest version of these notes you can find at http://mironchenko.com/index.php/en/iss-course

Please send me any corrections and/or comments, no matter how small, to the email:
andrii.mironchenko@uni-passau.de

Andrii Mironchenko
Odessa, Ukraine and Passau, Germany
December 10, 2015

Chapter 1

Introduction

1.1 Ordinary differential equations with a measurable right hand side

Consider the system of ordinary differential equations (ODEs) with external inputs

$$\begin{cases} \dot{x} = f(x,u), \ t > 0 \\ x(0) = x_0. \end{cases} \tag{1.1.1}$$

Here $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is measurable w.r.t. the second argument. We assume that inputs u belong to the space $\mathcal{U} = L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ of Lebesgue measurable essentially bounded functions. The norm in $\mathcal{U}$ is defined as

$$\|u\|_\infty := \operatorname*{ess\,sup}_{t \geq 0} |u(t)| = \inf_{D \subset \mathbb{R}_+,\ \mu(D)=0} \ \sup_{t \in \mathbb{R}_+ \setminus D} |u(t)|. \tag{1.1.2}$$

The norm of $x \in \mathbb{R}^d$ for any $d \in \mathbb{N}$ we introduce as $|x| := \sqrt{x_1^2 + \ldots + x_d^2}$.

Since the right-hand side of (1.1.1) is not continuous w.r.t. time, the system (1.1.1) does not possess in general classical continuously-differentiable solutions. Therefore we need to look for solutions with a lesser degree of smoothness.

Definition 1.1.1. *A function $f : [a,b] \to \mathbb{R}$ is called absolutely continuous, if for any $\varepsilon > 0$ there exists $\delta > 0$ so that for any finite r and any pairwise disjoint intervals (a_k, b_k), with $\sum_{k=1}^r |b_k - a_k| \leq \delta$ it follows that $\sum_{k=1}^r |f(b_k) - f(a_k)| \leq \varepsilon$*

Absolutely continuous functions are important for ODE theory since for these functions the Newton-Leibniz formula holds:

Theorem 1.1.1 (Theorem 3, p. 345, [50])**.** *Let $F \in AC(a,b)$. Then $\frac{d}{dt}F \in L_1(a,b)$ and for any $t \in [a,b]$ it holds that*

$$\int_a^t \frac{d}{ds} F(s) ds = F(t) - F(a). \tag{1.1.3}$$

The differentiability of absolutely continuous functions in a.e. sense motivates us to seek solutions of (1.1.1) is this class of functions:

Definition 1.1.2. *An absolutely continuous function $\phi : t \mapsto \phi(t,x,u)$ is called a solution of (1.1.1) corresponding to a given initial condition $x \in \mathbb{R}^n$ and to an input $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ if $\phi(0,x,u) = x$ holds and the equality (1.1.1) (after substitution of ϕ instead of x) holds almost everywhere.*

Remark 1.1.1. *Every absolutely continuous function is uniformly continuous (just set $n = 1$ in Definition 1.1.1). But not any uniformly continuous function is absolutely continuous. For example, consider a Cantor's stair function $f : [0, 1] \to [0, 1]$. This function is continuous on $[0, 1]$, and thus also uniformly continuous. It is also piecewise constant function almost everywhere on $[0, 1]$. Thus, $\frac{df}{dt} = 0$ almost everywhere, and hence $\int_0^1 \frac{df}{dt}(t)dt = 0$. However, $f(b) - f(a) = 1 - 0 = 1$, and thus f is not absolutely continuous.*

Integrating (1.1.1) from 0 to t and exploiting (1.1.3), we have that the solution ϕ of (1.1.1) satisfies

$$\phi(t, x, u) = x + \int_0^t f(\phi(s, x, u), u(s))ds. \tag{1.1.4}$$

Absolutely continuous solutions of (1.1.4) solve (1.1.1) in a.e. sense.

Having in mind the equation (1.1.4) one can develop existence and uniqueness theory for systems (1.1.1) with measurable inputs.

We call f Lipschitz continuous on bounded balls w.r.t. the first argument uniformly with respect to the second one if $\forall w > 0\ \exists L(w) > 0$, such that $\forall x, y : |x| \leq w,\ |y| \leq w$ and $\forall v \in \mathbb{R}^m$ it holds that

$$|f(y, v) - f(x, v)| \leq L(w)|y - x|. \tag{1.1.5}$$

Theorem 1.1.2. *Let f be Lipschitz w.r.t. the first argument uniformly w.r.t. the second one. Then for all $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and all initial conditions $x \in \mathbb{R}^n$ there exists (at least locally) the unique solution $\phi(\cdot, x, u)$ of (1.1.1).*

Proof. For the proof see [1, Paragraph 2.5].

□

Next we show that the flow map ϕ satisfies the semigroup property.

Proposition 1.1.3 (Semigroup property)**.** *Let the solution of (3.0.1) exists and is unique for any $x \in \mathbb{R}^n$ and any $u \in \mathcal{U}$. Then for all $x \in \mathbb{R}^n$, $u \in \mathcal{U}$ and all $t, \tau \geq 0$ we have*

$$\phi(t + \tau, x, u) = \phi(t, \phi(\tau, x, u), v). \tag{1.1.6}$$

where $v(r) = u(\tau + r)$, $\forall r \geq 0$.

Proof. Pick any initial condition $x \in \mathbb{R}^n$ and any input $u \in \mathcal{U}$. Pick also any $\tau \in (0, t)$ and define a shifted input $v \in \mathcal{U}$ by $v(t) = u(t + \tau)$, $t \geq 0$.

Due to (1.1.4) we have:

$$\begin{aligned}
\phi(t, x, u) &= x + \int_0^t f(\phi(s, x, u), u(s))ds \\
&= x + \int_0^\tau f(\phi(s, x, u), u(s))ds + \int_\tau^t f(\phi(s, x, u), u(s))ds \\
&= \phi(\tau, x, u) + \int_0^{t-\tau} f(\phi(s + \tau, x, v), v(s))ds \\
&= \phi(t - \tau, \phi(\tau, x, u), v)).
\end{aligned}$$

□

Besides existence and uniqueness of solutions a desired property of solutions is continuous dependence on initial states and inputs.

Definition 1.1.3. *We say that* (1.1.1) *depends continuously on inputs and on initial states, if* $\forall x \in \mathbb{R}^n, \forall u \in \mathcal{U}, \forall \tau > 0$ *and* $\forall \varepsilon > 0$ *there exist* $\delta = \delta(x,u,\tau,\varepsilon) > 0$*, such that* $\forall x' \in \mathbb{R}^n : |x - x'| < \delta$ *and* $\forall u' \in \mathcal{U} : \|u - u'\|_\infty < \delta$ *it holds*

$$|\phi(t,x,u) - \phi(t,x',u')| < \varepsilon, \quad \forall t \in [0,\tau].$$

By imposing stronger conditions on nonlinearity f, than those in Theorem 1.1.2, one can prove

Theorem 1.1.4. *Let* $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *satisfy the following condition: there exist* $q \in \mathcal{K}_\infty$*:* $\forall C > 0\ \exists L(C) > 0$*, such that* $\forall x_1, x_2 : |x_1| \le C,\ |x_2| \le C$, $\forall u_1, u_2 \in \mathbb{R}^m :\ |u_1| \le C,\ |u_2| \le C$ *it holds that*

$$\begin{aligned} |f(x_1,&u_1) - f(x_2,u_2)| \\ &\le L(C)\big(|x_1 - x_2| + q(|u_1 - u_2|)\big). \end{aligned} \tag{1.1.7}$$

Then (1.1.1) *depends continuously on inputs and on initial states.*

Assumption 1.1.1. *We suppose throughout the paper that:*

(i) f *is Lipschitz continuous on bounded subsets of* $\mathbb{R}^n$*, uniformly with respect to the second argument.*

(ii) $f(x,\cdot)$ *is continuous for all* $x \in \mathbb{R}^n$*.*

(iii) The solutions of (1.1.1) *depend continuously on initial states and inputs.*

Exercise 1.1.1. *Consider an equation*

$$\dot{x}(t) = x^{1/3}(t)$$

subject to initial condition $x(0) = 0$*. Clearly,* $x \equiv 0$ *is a solution of this equation. Find by means of a separation of variables another solution of this equation. Can you find more solutions?*

Exercise 1.1.2. *Investigate existence and uniqueness of solutions of the problem*

$$\dot{x} = f(x), \quad x(0) = x_0, \quad f(x) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x \ge 0 \end{cases}$$

for all $x \in \mathbb{R}$*. If the solution exists, then what is its maximal interval of existence?*

1.2 Comparison functions

In this section we introduce the following classes of comparison functions which simplify greatly the formulation of stability properties for the system (1.1.1):

$$\begin{aligned}
\mathcal{P} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and } \gamma(r) > 0 \text{ for } r > 0\} \\
\mathcal{K} &:= \{\gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing}\} \\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\
\mathcal{L} &:= \Big\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \,\Big|\, \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\Big\} \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot,t) \in \mathcal{K},\ \forall t \ge 0,\ \beta(r,\cdot) \in \mathcal{L},\ \forall r > 0\}
\end{aligned}$$

Functions of class $\mathcal{P}$ are also called positive definite functions.

Let id be an identity operator on $\mathbb{R}_+$.

Proposition 1.2.1. *The following properties hold*

1. *For all* $f, g \in \mathcal{K}$ *it follows that* $f \circ g \in \mathcal{K}$*.*
2. *For any* $f \in \mathcal{K}_\infty$ *there exists* f^{-1}*, which also belongs to* $\mathcal{K}_\infty$*.*

3. *For any $f \in \mathcal{K}$, $g \in \mathcal{L}$ it holds that $f \circ g \in \mathcal{L}$ and $g \circ f \in \mathcal{L}$.*

Proof. 1. Since f, g are continuous, $f \circ g$ is also continuous and defined over $\mathbb{R}_+$. Due to $f(0) = g(0) = 0$ it holds also $f(g(0)) = 0$. The monotonicity of $f \circ g$ is clear.

2. Since $f \in \mathcal{K}_\infty$, it is a bijection on the space $\mathbb{R}_+$. Thus, there exists an inverse map $f^{-1} \in \mathcal{K}_\infty$. For any $r_1 < r_2$ we have $f(f^{-1}(r_1)) < f(f^{-1}(r_2))$. Since f is monotonically increasing, it must hold $f^{-1}(r_1) < f^{-1}(r_2)$.

3. Clear.

□

Proposition 1.2.2. ***Weak triangle inequality***. *For any $\gamma \in \mathcal{K}$, and any $\sigma \in \mathcal{K}_\infty$ it holds that:*

$$\gamma(a + b) \leq \max\{\gamma(a + \sigma(a)), \gamma(b + \sigma^{-1}(b))\}. \tag{1.2.1}$$

Proof. Pick any $\sigma \in \mathcal{K}_\infty$. Then either $b \leq \sigma(a)$ or $a \leq \sigma^{-1}(b)$ holds. This immediately implies (1.2.1). □

In particular, setting $\sigma := id$ in (1.2.1) we obtain for any $\gamma \in \mathcal{K}_\infty$ a simple inequality

$$\gamma(a + b) \leq \max\{\gamma(2a), \gamma(2b)\}.$$

An important result on comparison functions which simplifies the proofs of a number of results in ISS theory, is a so-called Sontag's $\mathcal{KL}$-Lemma, originally proved in [69, Proposition 7] (see also [47]):

Proposition 1.2.3. *For each $\beta \in \mathcal{KL}$ and any $\lambda > 0$ there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$:*

$$\alpha_1(\beta(s, t)) \leq \alpha_2(s) e^{-\lambda t}, \quad t, s \in \mathbb{R}_+. \tag{1.2.2}$$

Proof. We follow the lines of [74, Lemma 3, p.326].

First pick the functions $\rho \in \mathcal{K}_\infty$ and $\theta \in \mathcal{L}$ so that

$$\beta(\rho(t), t) \leq \theta(t) \quad \forall t \geq 0. \tag{1.2.3}$$

Next we show that such functions exist.

First, it is clear that for all $r > 0$ and any $q \in (0, \beta(r, 0)]$ there exists a unique $t \geq 0$ so that $\beta(r, t) = q$. Pick two sequences $\{q_i\}_{i=1}^\infty$: $q_i \to 0$, $i \to \infty$ and $\{r_i\}_{i=1}^\infty$: $r_i \to \infty$ together with a corresponding sequence $\{\tilde{t}_i\}_{i=1}^\infty$ so that for all $i = 1, \ldots, \infty$ we have

$$\beta(r_i, \tilde{t}_i) = q_i.$$

Pick now any sequence $\{t_i\}_{i=1}^\infty$: t_i is strictly increasing, $t_i \to \infty$ as long as $i \to \infty$ and $t_i \geq \tilde{t}_i$ for all $i = 1, \ldots, \infty$. Define also $t_0 := 0$. Since $\beta \in \mathcal{KL}$ we have

$$\beta(r_i, t_i) \leq \beta(r_i, \tilde{t}_i) = q_i.$$

Now define θ so that $\theta(t_i) = q_{i-1}$ for all $i = 1, \ldots, \infty$ and enlarge the domain of definition of θ so that $\theta \in \mathcal{L}$. Also define $\rho(t_i) = r_{i-1}$ for all $i = 1, \ldots, \infty$ where we define $r_0 := r_1/2$. And again enlarge the domain of definition of ρ so that $\rho \in \mathcal{K}_\infty$.

By construction we have for any $i = 0, 1, \ldots, \infty$ and for all $t \in [t_i, t_{i+1}]$ it holds that

$$\beta(\rho(t), t) \leq \beta(\rho(t_{i+1}), t_i) = \beta(r_i, t_i) \leq q_i = \theta(t_{i+1}) < \theta(t).$$

This shows that claim.

Denote the inverse of θ as θ^{-1}, which is defined, continuous and strictly decreasing on the interval $(0, \theta(0)]$. Pick any $\lambda > 0$. Then the function $s \mapsto e^{-2\lambda\theta^{-1}(s)}$ is defined for $s \in (0, \theta(0)]$ and is continuous and strictly increasing on its domain of definition. Setting $e^{-2\lambda\theta^{-1}(0)} := 0$, we extend this function by continuity to $[0, \theta(0)]$.

Obviously, there exist $\alpha_1 \in \mathcal{K}_\infty$ so that for all $s \in [0, \theta(0)]$ it holds that

$$\alpha_1(s) \leq e^{-2\lambda\theta^{-1}(s)}. \tag{1.2.4}$$

Substituting $s := \theta(t)$ in this inequality and using (1.2.3) we derive that

$$\alpha_1(\beta(\rho(t), t))e^{2\lambda t} \leq \alpha_1(\theta(t))e^{2\lambda t} \leq 1 \quad \forall t \geq 0. \tag{1.2.5}$$

Using (1.2.5) and the fact that $\beta \in \mathcal{KL}$, we obtain that for $s, t \in \mathbb{R}_+ : \ 0 < s \leq \rho(t)$

$$\begin{aligned} \alpha_1(\beta(s,t))e^{\lambda t} &= \sqrt{\alpha_1(\beta(s,0))}\frac{\sqrt{\alpha_1(\beta(s,t))}}{\sqrt{\alpha_1(\beta(s,0))}}\sqrt{\alpha_1(\beta(s,t))e^{2\lambda t}} \\ &\leq \sqrt{\alpha_1(\beta(s,0))}\sqrt{\alpha_1(\beta(\rho(t),t))e^{2\lambda t}} \\ &\leq \sqrt{\alpha_1(\beta(s,0))} \end{aligned} \tag{1.2.6}$$

On the other hand, for $s \geq \rho(t)$ it holds that

$$\alpha_1(\beta(s,t))e^{\lambda t} \leq \alpha_1(\beta(s,0))e^{\lambda\rho^{-1}(s)}. \tag{1.2.7}$$

Pick any $\alpha_2 \in \mathcal{K}_\infty$: for all $s \geq 0$

$$\alpha_2(s) \geq \max\{\alpha_1(\beta(s,0))e^{\lambda\rho^{-1}(s)}, \sqrt{\alpha_1(\beta(s,0))}\}.$$

This implies that (1.2.2) holds for all $s, t \geq 0$. □

We will need in the future a trivial restatement of Proposition 1.2.3:

Corollary 1.2.4. *For each $\beta \in \mathcal{KL}$ and any $\lambda > 0$ there exist $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$:*

$$\beta(s,t) \leq \tilde{\alpha}_1(\tilde{\alpha}_2(s)e^{\lambda t}) \quad \forall s, t \geq 0. \tag{1.2.8}$$

Several other useful properties of comparison functions you can find in exercises:

Exercise 1.2.1. ***Another weak triangle inequality***. *Show that for any $\gamma \in \mathcal{K}$, and any $\rho \in \mathcal{K}_\infty$ so that $\rho - id \in \mathcal{K}_\infty$ and for any $a, b \in \mathbb{R}_+$ it holds that:*

$$\gamma(a+b) \leq \gamma \circ \rho(a) + \gamma \circ \rho \circ (\rho - id)^{-1}(b).$$

Exercise 1.2.2. *Let $\beta \in \mathcal{KL}$. Prove that:*

- *For all $r > 0$ there exist $\sigma \in \mathcal{K}_\infty$ and $\zeta \in \mathcal{L}$ so that*

$$\beta(|x|, t) \leq \sigma(|x|)\zeta(t), \tag{1.2.9}$$

  for all $t \geq 0$ and all $x \in \mathbb{R}^n$: $|x| \leq r$.

Exercise 1.2.3. ***$\mathcal{K}_\infty$-inequality***. *Prove that for all $a, b \geq 0$, for all $\alpha \in \mathcal{K}_\infty$ and all $r > 0$ it holds that*

$$ab \leq ra\alpha(a) + b\alpha^{-1}(\tfrac{b}{r}). \tag{1.2.10}$$

Exercise 1.2.4. ***Lower bounds of $\mathcal{P}$-functions***. *Prove that for any positive definite function $\rho : \mathbb{R}^n \to \mathbb{R}_+$ (i.e. $\rho(0) = 0$ and $\rho(x) > 0$ for $x \neq 0$) there exist $\alpha \in \mathcal{K}_\infty$ and $\zeta \in \mathcal{L}$: for any $r \in \mathbb{R}_+$ the following holds:*

$$\rho(x) \geq \alpha(|x|)\zeta(|x|). \tag{1.2.11}$$

1.3 Some results from analysis

In this section we collect several results from analysis which will be used in these notes

Proposition 1.3.1 (Rademacher's theorem). *Let $f : U \to \mathbb{R}^m$, $U \subset \mathbb{R}^m$ be locally Lipschitz continuous in U. Then f is differentiable almost everywhere in U.*

1.4 Some lemmas from analysis

We prove the following useful lemma

Lemma 1.4.1. *Let (R_X, ρ_X) and (R_Y, ρ_Y) be metric spaces, and $X \subset R_X$ be a compact set, $Y \subset R_Y$. Assume, that $f : X \times Y \mapsto \mathbb{R}$ is continuous on $X \times Y$. Then function $g(y) = \max_{x \in X} f(x, y)$ is continuous on Y.*

Proof. Denote by $X_y := \arg\max_{x \in X} f(x, y)$ - the set of values $x \in X$, on which a function $f(\cdot, y)$ takes its maximum.

At first we are going to prove auxiliary statement, namely $\forall \omega > 0$ $\exists \delta > 0$, such that

$$\forall y \in Y\colon \rho_Y(y, y_0) < \delta \Rightarrow X_y \subset U_\omega(X_{y_0}) = \bigcup_{x \in X_{y_0}} U_\omega(x),$$

where $U_\omega(x)$ is a ball with centre in x and radius ω.

Assume, that this statement does not hold. Then there exists $\omega_0 > 0$ and sequences $\{\delta_n\}_{n=1}^{n=\infty}$, $\lim_{n\to\infty} \delta_n = 0$, $\{y_n\}_{n=1}^{n=\infty} \subset Y$, $\rho_Y(y_n, y_0) < \delta_n$ and $\{x_n\}_{n=1}^{n=\infty}$, where $x_n \in X_{y_n} \backslash U_{\omega_0}(X_{y_0})$.

By construction $\lim_{n\to\infty} y_n = y_0$. X is compact, therefore $\{x_n\}_{n=1}^{n=\infty}$ is bounded, and from Bolzano-Weierstrass theorem it follows, that a convergent subsequence $\{x_{n_k}\}_{k=1}^{k=\infty} \subset \{x_n\}_{n=1}^{n=\infty}$ exists. Let $\lim_{k\to\infty}(x_{n_k}, y_{n_k}) = (x^*, y_0) \in X \times Y$.

If $x^* \in X_{y_0}$, then some elements of $\{x_{n_k}\}_{k=1}^{k=\infty}$ belong to $U_{\omega_0}(x^*) \subset U_{\omega_0}(X_{y_0})$, and we have a contradiction.

Let $x^* \notin X_{y_0}$. Then $f(x^*, y_0) < f(x_0, y_0)$ for some $x_0 \in X_{y_0}$, and therefore disjunct balls $U_s((x^*, y_0)) \subset X \times Y$ and $U_s((x_0, y_0)) \subset X \times Y$ for some $s > 0$ exist, such that $\forall (x', y') \in U_s((x^*, y_0))$, $\forall (x, y) \in U_s((x_0, y_0))$ it holds $f(x', y') < f(x, y)$. But $\forall s > 0$ in $U_s((x^*, y_0))$ infinitely many elements of the sequence (x_{n_k}, y_{n_k}) exist. Let one of them be $(x_{n_{k1}}, y_{n_{k1}})$. We have, that function $f(\cdot, y_{n_{k1}})$ does not possess maximum at $x_{n_{k1}}$, and therefore $x_{n_{k1}} \notin X_{y_{n_{k1}}}$, and we come to a contradiction. Our statement is proven.

Now we can prove the claim of the lemma. We have, that $\forall \omega > 0$ $\exists \delta_1 > 0$, such that $\forall y \in Y$ with $\rho_Y(y, y_0) < \delta_1$ there exist $x_1 \in X_y$, $x_0 \in X_{y_0}$ with $\rho_X(x_1, x_0) < \omega$, and we have

$$\begin{aligned}
&|g(y) - g(y_0)| = |\max_{x \in X} f(x, y) - \max_{x \in X} f(x, y_0)| \\
&= |f(x_1, y) - f(x_0, y_0)| \\
&\le |f(x_1, y) - f(x_1, y_0)| + |f(x_1, y_0) - f(x_0, y_0)|.
\end{aligned}$$

From the continuity of $f(x_1, \cdot)$ at the point y_0 we have, that $\forall \varepsilon_2 > 0$ $\exists \delta_2 : \forall y \in Y : \rho_Y(y, y_0) < \delta_2 \Rightarrow |f(x_1, y) - f(x_1, y_0)| < \varepsilon_2$.

To estimate $|f(x_1, y_0) - f(x_0, y_0)|$ we use that $\forall y_0 \in Y$ function $f(\cdot, y_0)$ is a continuous function, defined on a compact set, and, according to Heine-Cantor theorem, it is uniformly continuous, that is $\forall \varepsilon_1 > 0$ $\exists \omega : \forall x_1, x_0 \in X : \rho_X(x_1, x_0) < \omega \Rightarrow |f(x_1, y_0) - f(x_0, y_0)| < \varepsilon_1$.

Taking $\delta = \min\{\delta_1, \delta_2\}$, and $\varepsilon = \varepsilon_1 + \varepsilon_2$, we have, that $\forall \varepsilon > 0$ $\exists \delta > 0$ such that

$$\forall y \in Y : \rho_Y(y, y_0) < \delta \Rightarrow |g(y) - g(y_0)| < \varepsilon.$$

Thus, g is continuous at y_0. Since y_0 was chosen arbitrarily, g is continuous in all Y. □

Lemma 1.4.2. *Let $g : \mathbb{R}^n \to \mathbb{R}_+$ be a continuous function and $g(0) = 0$. Then $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\psi_1(r) := \inf_{|x|\geq r} g(x)$ and $\psi_2(r) := \sup_{|x|\leq r} g(x)$ are well-defined, continuous, nondecreasing and $\psi_1(0) = \psi_2(0) = 0$.*

Proof. Clearly, ψ_1, ψ_2 are well-defined, nondecreasing and $\psi_1(0) = \psi_2(0) = 0$. Let ψ_2 be discontinuous at some r_1. Since the supremum is taken over a compact set, then the set $S_r := \{x \in \mathbb{R}^n : |x| \leq r,\ \psi_2(x) = \psi_2(r)\}$ (the set of all points at which a supremum is achieved) is nonempty.

Analogously as in Lemma 1.4.1 we obtain that $\forall \omega > 0\ \exists \delta > 0$, such that

$$\forall r : |r - r_1| < \delta \Rightarrow S_r \subset U_\omega(S_{r_1}) = \bigcup_{x \in S_{r_1}} U_\omega(x),$$

where $U_\omega(x)$ is a ball with centre in x and radius ω.

Using argument, similar to those in Lemma 1.4.1 we can see that discontinuity of ψ_2 contradicts to the continuity of f. □

Lemma 1.4.3. *Let $\{x_k^1\}_{k=1}^\infty$, ..., $\{x_k^m\}_{k=1}^\infty$ be sequences of real numbers. Let the limit $\lim_{k\to\infty} \max_{1\leq i\leq m}\{x_k^i\}$ exist. Then it holds that*

$$\lim_{k\to\infty} \max_{1\leq i\leq m} \{x_k^i\} = \max_{1\leq i\leq m} \{\overline{\lim_{k\to\infty}}\, x_k^i\}, \tag{1.4.1}$$

where $\overline{\lim}_{k\to\infty} x_k^i$ is the upper limit of the sequence x_k^i.

Proof. For all $k \in \mathbb{N}$ define $i(k) = \arg\max_{1\leq i\leq m}\{x_k^i\}$ - the index of the maximal element of $\{x_k^i\}$, $i = 1, \ldots, n$ (if there are more than one maximal element, than take arbitrary index). Then $\max_{1\leq i\leq m} x_k^i = x_k^{i(k)}$ for all $k \in \mathbb{N}$. Extract from the sequence $\{x_k^{i(k)}\}$ the maximal subsequences of the form $\{x^j_{n_k^j}\}$, $j = 1, \ldots, m$, where n_k^j is the monotone increasing sequence of indexes. At least some of $\{x^j_{n_k^j}\}$, $j = 1, \ldots, m$ are infinite (without loss of generality let it be $\{x^1_{n_k^1}\}$).

The sequence $\{x_k^{i(k)}\}$ is convergent, hence all its subsequences are convergent and have the same limit value. Thus we obtain

$$\lim_{k\to\infty} \max_{1\leq i\leq m} \{x_k^i\} = \lim_{k\to\infty} x_k^{i(k)} = \lim_{k\to\infty} x^1_{n_k^1} \leq \overline{\lim_{k\to\infty}}\, x_k^1 \leq \max_{1\leq i\leq m} \overline{\lim_{k\to\infty}}\, x_k^i. \tag{1.4.2}$$

To obtain the reverse inequality, take any sequence $\{x^i_{n_k}\}$, such that

$$\lim_{k\to\infty} x^i_{n_k} = \max_{1\leq i\leq m} \overline{\lim_{k\to\infty}}\, x_k^i.$$

We have that $\max_{1\leq i\leq m}\{x^i_{n_k}\} \geq x^i_{n_k}$, and so

$$\lim_{k\to\infty} \max_{1\leq i\leq m} \{x_k^i\} \geq \max_{1\leq i\leq m} \{\overline{\lim_{k\to\infty}}\, x_k^i\}. \tag{1.4.3}$$

From (1.4.2) and (1.4.3) we obtain (1.4.1). □

Corollary 1.4.1. *Let $f_i : \mathbb{R} \to \mathbb{R}$ are defined and bounded in some neighborhood D of $t = 0$. Then it holds*

$$\overline{\lim_{t\to 0}} \max_{1\leq i\leq m} \{f_i(t)\} = \max_{1\leq i\leq m} \{\overline{\lim_{t\to 0}} f_i(t)\} \tag{1.4.4}$$

Proof. Under made assumptions the upper limits in both parts of the equation (1.4.4) exist. From $\max_{1\leq i\leq m}\{f_i(t)\} \geq f_i(t)\ \forall i = 1, \ldots, m$, for all $t \in D$. Thus,

$$\overline{\lim_{t\to 0}} \max_{1\leq i\leq m} \{f_i(t)\} \geq \max_{1\leq i\leq m} \{\overline{\lim_{t\to 0}} f_i(t)\}$$

To prove the converse inequality, we use Lemma 1.4.3.

$$\overline{\lim_{t\to 0}} \max_{1\leq i\leq m} \{f_i(t)\} = \sup_{t_{n_k}\to 0} \lim_{k\to\infty} \max_{1\leq i\leq m} \{f_i(t_{n_k})\}$$

$$= \sup_{t_{n_k}\to 0} \max_{1\leq i\leq m} \{\overline{\lim_{k\to\infty}} f_i(t_{n_k})\} \leq \max_{1\leq i\leq m} \{\overline{\lim_{t\to 0}} f_i(t)\},$$

where the sup is taken over all convergent to 0 sequences t_{n_k}. □

1.5 Concluding remarks

Existence and uniqueness theory for the ODEs with a right-hand side, which is measurable in t, has been developed by Caratheodory. For much more on this theory you can consult, e.g. [13].

Classes of comparison functions have been introduced by Jose Massera in 1950s and were widely used by Wolfgang Hahn in his book [26] to characterize stability notions of the systems. The usefulness of these classes of functions for the ISS framework made their use standard within the systems theoretic society. For much more on comparison functions you can consult [47].

Chapter 2

Stability of undisturbed systems

Before we proceed to our main subject, which is input-to-state stability of control systems, it is rewarding to look at the classical stability theory for autonomous ODE systems of the form

$$\dot{x} = f(x), \tag{2.0.1}$$

where $x(t) \in \mathbb{R}^n$ and f is locally Lipschitz. Our main aim is to derive characterizations of global asymptotic stability of the system (2.0.1) in terms of comparison functions, uniform attraction times and Lyapunov functions.

By $\phi(t, x)$ we denote the state of the system (2.0.1) at time t if at time 0 its state was equal to x.

Definition 2.0.1. *A point $y \in \mathbb{R}^n$ is called an equilibrium (or stationary point) of* (2.0.1) *if $\phi(t, y) = y$ for all $t \geq 0$.*

In other words, y is an equilibrium of (2.0.1) if and only if $f(y) = 0$.

2.1 Basic definitions

Denote $B_r(y) := \{x \in \mathbb{R}^n : |x - y| \leq r\}$ - a closed ball in $\mathbb{R}^n$ of radius r around $y \in \mathbb{R}^n$. For short we denote $B_r := B_r(0)$.

Next we define the stability notions which we investigate in this section.

Definition 2.1.1. *An equilibrium x^* of the system* (2.0.1) *is called*

- *locally stable (LS), if for any $\varepsilon > 0$ there exists $\delta > 0$ so that $\forall x \in B_\delta(x^*)$ it follows that $|\phi(t, x) - x^*| < \varepsilon$, for all $t \geq 0$.*
- *locally attractive (ATT) if there exists $r > 0$ so that for all $x \in B_r(x^*)$ it follows that $\lim_{t\to\infty} |\phi(t, x) - x^*| = 0$.*
- *globally attractive (GATT) if $\lim_{t\to\infty} |\phi(t, x) - x^*| = 0$ for all $x \in \mathbb{R}^n$.*
- *locally asymptotically stable (AS), if* (2.0.1) *is locally stable and locally attractive.*
- *globally asymptotically stable (GAS), if* (2.0.1) *is locally stable and globally attractive.*

In control theory it is usual to call the system (2.0.1) stable/asymptotically stable if the origin is a stable/asymptotically stable equilibrium for (2.0.1). In these notes we adopt this convention.

The graphs of typical stable and asymptotically stable systems are depicted in Figures 2.1, 2.2.

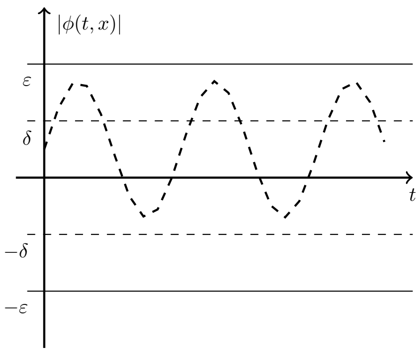

Figure 2.1: Typical trajectory of a stable system

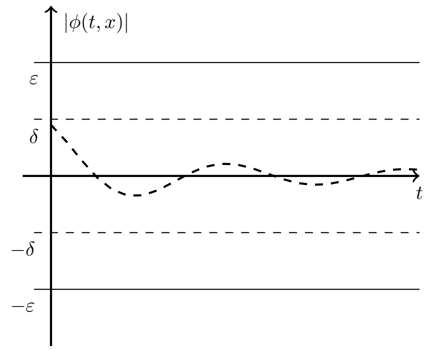

Figure 2.2: Typical trajectory of an asymptotically stable system

Remark 2.1.1. *It is useful to compare the notion of the local stability to the notion of a continuous dependence of solutions w.r.t. initial data (at $x = 0$), which states that for any $T > 0$ and any $\varepsilon > 0$ we can find sufficiently small $\delta = \delta(\varepsilon, T) > 0$ so that $\forall x \in B_\delta$ and for $t \in [0, T]$ it holds that $|\phi(t, x)| < \varepsilon$. The local stability is a much stronger property, which ensures that the trajectory stays within the ε-neighborhood of a stationary point for all times $t \geq 0$.*

In general, investigation of stability of nonlinear dynamical systems is an important and nontrivial task. However, if $x(t)$ is a one-dimensional vector, the problem can be easily analyzed in full generality, which will be done in the next example.

Example 2.1.1. ***Stability of 1-dimensional ($x \in \mathbb{R}$) systems*** (2.0.1). *Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, $f(a) = f(b) = 0$ for some $a < b$ and let $f(x) \neq 0$ for all $x \in (a, b)$. We are going to show that for any $x_0 \in (a, b)$ the solution of (2.0.1) exists for all times $t \in \mathbb{R}_+$ and $\lim_{t\to\infty} x(t) = c$ with $c \in \{a, b\}$.*

Since we have assumed that $f(x) \neq 0$ for all $x \in (a, b)$, then due to continuity of f, either $f(x) > 0$ for all $x \in (a, b)$ or $f(x) < 0$ for all $x \in (a, b)$. Assume that $f(x) > 0$ for $x \in (a, b)$. The other case can be treated analogously.

Pick any $x_0 \in (a, b)$. Since f is Lipschitz on $\mathbb{R}$, there exists exactly one solution of (2.0.1) with initial condition x_0 on some maximal interval I. Since $f(x) > 0$ for all $x \in (a, b)$ this solution is monotonously increasing. Assume that there exists $t^* \in I$ so that $\phi(t^*, x_0) = b$ holds. Then there would be two distinct solutions emanating from the initial condition $x(0) = b$, which leads to contradiction since f is Lipschitz continuous.

Thus, the solution $\phi(\cdot, x_0)$ is defined for all $t \in \mathbb{R}_+$ and $\phi(t, x_0) \leq b$ for all $t \in \mathbb{R}_+$. Since $\phi(\cdot, x_0)$ is monotonically increasing, there exists $\lim_{t\to\infty} \phi(t, x_0) =: c$.

Due to equation (1.1.4) it holds $\phi(t, x_0) - x_0 = \int_0^t f(\phi(s, x_0))ds$. If $f(c) \neq 0$, then taking limits of this equality when $t \to \infty$ leads to contradiction (the right-hand side is infinite, while the left hand side is finite). Thus, $f(c) = 0$ which implies that $c = b$.

Analogously, if $f(x) < 0$ for all $x \in (a, b)$, then $\phi(t, x_0) \to a$ as long as $t \to \infty$. Thus, in order to study stability of one-dimensional systems one can find the set $S := \{x \in \mathbb{R} : f(x) = 0\}$ and apply above considerations.

Local attractivity and local stability are conceptually different notions in the sense that neither of them implies another one. It is easy to find a stable non-attractive system (just consider $\dot{x} = 0$). Next we show an example of a system which possesses an equilibrium, which attracts all the points of $\mathbb{R}^2 \backslash \{0\}$ but is not locally stable (taken from [61, p. 119]).

Example 2.1.2. *Consider the following system, defined in polar coordinates.*

$$\begin{aligned} \dot{r} &= r(1 - r), \\ \dot{\theta} &= \sin^2(\theta/2). \end{aligned} \tag{2.1.1}$$

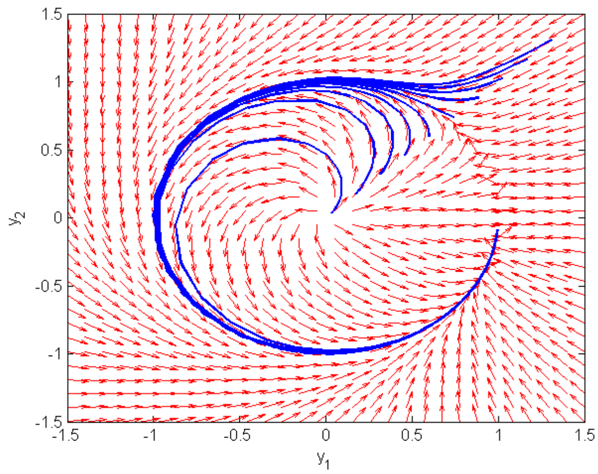

Figure 2.3: Velocity field and some of trajectories of (2.1.1)

This system has two equilibrium points: $(0,0)$ *and* $(1,0)$. *It is easy to see that the equilibrium* $(0,0)$ *is unstable, and also it does not attract any trajectory of the system* (2.1.1). *At the same time the point* $(1,0)$ *attracts the trajectories, starting in any point* $x_0 \in \mathbb{R}^2 \backslash \{0\}$. *However, the point* $(1,0)$ *is not stable, since there are points starting arbitrarily close to* $(1,0)$, *which go around the unit circle before they converge to* $(1,0)$.

A more complicated example of a globally attractive system which is unstable was proposed by Vinograd [80]:

$$\begin{aligned} \dot{x} &= \frac{x^2(y-x)+y^5}{r^2(1+r^4)}, \\ \dot{y} &= \frac{y^2(y-2x)}{r^2(1+r^4)}, \end{aligned} \tag{2.1.2}$$

where $r^2 = x^2 + y^2$.

The equilibrium $(x,y) = (0,0)$ is globally attractive, but is not stable. The phase portrait of this system is given by Figure 2.1. For an analysis of this system you may consult [61, p.120].

Exercise 2.1.1. *System* (2.0.1) *is called globally nonuniformly stable at zero if for all* $x \in \mathbb{R}^n$ *it holds that*

$$\sup_{t \geq 0} |\phi(t,x)| < \infty.$$

Prove that GATT implies global nonuniform stability of (2.0.1).

Exercise 2.1.2. (2.0.1) *is LS* $\Leftrightarrow$ *the function* $g : x \mapsto \sup_{t \geq 0} |\phi(t,x)|$ *is well-defined in a certain neighborhood of* 0 *and is continuous at the origin.*

Exercise 2.1.3. *Let* $A \in \mathbb{R}^{n \times n}$. *Consider a linear system*

$$\dot{x} = Ax. \tag{2.1.3}$$

Prove that for (2.1.3)

1. *ATT implies LS*
2. *ATT = GATT (and consequently AS = GAS).*

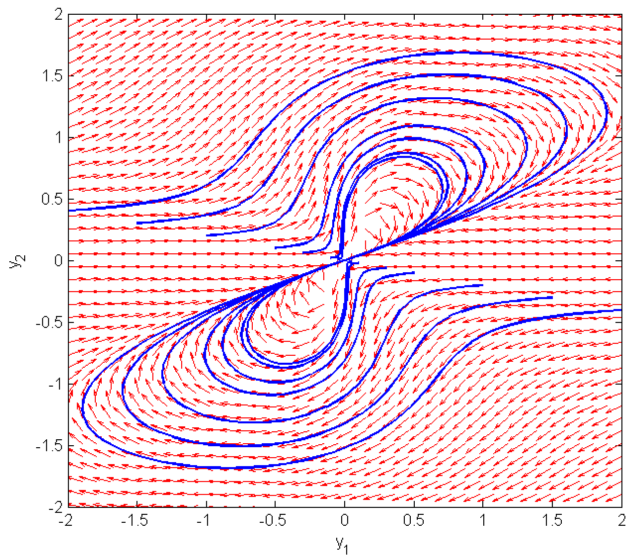

Figure 2.4: Velocity field and some of trajectories of (2.1.2)

Exercise 2.1.4. ***Theorem***. *The equilibrium $x \equiv 0$ is globally attractive for the system $\dot{x} = x^2$, $x(t) \in \mathbb{R}$.*

Proof. *The general soluion of $\dot{x} = x^2$ is given by $x(t) = -\frac{1}{t+c}$.*

It follows that $\forall c \in \mathbb{R}$ it holds that $\lim_{t\to\infty} -\frac{1}{t+c} = 0$. Thus, every solution coverges to 0. ■

But strangely, $\dot{x} > 0$ for all $x > 0$. Why?

2.2 Characterizations of GAS

In this section we derive several characterizations of GAS property, which give us more insights about the nature of GAS systems.

We start with a simple lemma

Lemma 2.2.1. (2.0.1) *is LS $\Leftrightarrow$ $\exists \sigma \in \mathcal{K}$ and some $r > 0$ so that $\forall x \in B_r$*

$$|\phi(t, x)| \leq \sigma(|x|). \tag{2.2.1}$$

Proof. If the system is LS, then for all $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$: for all $x \in B_\delta$ and for all $t \geq 0$

$$|\phi(t, x)| \leq \varepsilon.$$

Without loss of generality we assume that δ is continuous and increasing as a function of ε. Since 0 is an equilibrium of (2.0.1), $\delta(0) := 0$. Thus, $\delta \in \mathcal{K}$. Define $\sigma(s) := \delta^{-1}(s)$ for $s \in [0, a]$, where $a = \delta(+\infty)/2$. We can enlarge this σ to the class $\mathcal{K}$ function on the whole $\mathbb{R}_+$. Then (2.2.1) holds with this σ.

The converse statement follows by setting $\delta(\varepsilon) := \sigma^{-1}(\varepsilon)$ for all $\varepsilon > 0$. □

Lemma 2.2.1 tells us that local stability is a 'uniform' notion in the sense that for all initial values with the same norm we can find a common upper bound for the maximal deviation of trajectories from the origin. In contrast to this, ATT and GATT are not uniform notions. All trajectories of a GATT system converge to the origin, but the rate of convergence to the origin may vary drastically for the initial values with the same norm. Next we introduce the uniform counterparts of GATT and GAS notions. The relationship between nonuniform and uniform notions will be the main topic in this section.

Definition 2.2.1. *The system* (2.0.1) *is called*

- *globally stable (GS), if there exists* $\sigma \in \mathcal{K}_\infty$ *so that for all* $t \geq 0$ *and for all* $x \in \mathbb{R}^n$ *we have*

$$|\phi(t,x)| \leq \sigma(|x|). \tag{2.2.2}$$

- *uniformly globally attractive (UGATT), if for all* $\varepsilon, \delta > 0$ *there exists* $T = T(\varepsilon, \delta) < \infty$ *so that*

$$\forall t \geq T, \ \forall x \in B_\delta \quad \Rightarrow \quad |\phi(t,x)| \leq \varepsilon. \tag{2.2.3}$$

- *globally asymptotically stable at zero uniformly with respect to state (UGASs), if there is a* $\beta \in \mathcal{KL}$*, such that for all* $\phi_0 \in \mathbb{R}^n$ *and all* $t \geq 0$ *the following holds*

$$|\phi(t,\phi_0)| \leq \beta(|\phi_0|, t). \tag{2.2.4}$$

Now we can prove

Theorem 2.2.1. *Let the solutions of* (2.0.1) *depend continuously on initial values.*
The following notions are equivalent for the system (2.0.1)*:*

(i) UGAS

(ii) UGATT

(iii) GAS

Proof. (UGAS $\Rightarrow$ UGATT). Let (2.0.1) be UGASs. Take arbitrary $\varepsilon, \delta > 0$. Define $T = T(\varepsilon, \delta)$ as a solution of the equation $\beta(\delta, T) = \varepsilon$ (if it exists, then it is unique, because of monotonicity of β w.r.t. the second argument, if it does not exist, we put $T(\varepsilon, \delta) = 0$). Then for all $t \geq T$, $|x| \leq \delta$

$$|\phi(t,x)| \leq \beta(|x|, t) \leq \beta(|x|, T) \leq \varepsilon,$$

and the estimate (2.2.3) holds.

(UGATT $\Rightarrow$ GAS). Clearly, UGATT implies GATT. Let us prove that UGATT implies LS.

Pick any $\varepsilon, \delta > 0$. Since (2.0.1) is UGATT, there exist $T = T(\varepsilon, \delta)$: for all $x \in B_\delta$ it follows $|\phi(t,x)| < \varepsilon$ for all $t \geq T$. On the other hand, since (2.0.1) depends continuously on initial states, there exists $\delta_2 < \delta_1$: for all $x \in B_{\delta_2}$, $\forall t \in [0,T]$

$$|\phi(t,x) - \phi(t,0)| = |\phi(t,x)| \leq \varepsilon.$$

Then for all $x \in B_{\delta_2}$ and for all $t \geq 0$

$$|\phi(t,x)| \leq \varepsilon.$$

Thus (2.0.1) is LS. Overall, UGATT implies GATT+LS, which is equivalent to GAS.

(GAS $\Rightarrow$ UGAS).

TO DO □

Remark 2.2.2. *We have already seen in Example 2.1.1 that GATT does not imply LS, because the solutions can move far away before they converge to the origin. Theorem 2.2.1 shows another property of GATT systems which are not LS: their time of convergence is not uniform for the initial states with the same norm.*

The equivalence between GAS and UGAS does not hold anymore for infinite-dimensional systems. We give next a simple example of this fact.

Example 2.2.2. *Consider an infinite ODE system*

$$\dot{x}_k = -\frac{1}{k}x_k, \quad k = 1, \dots, \infty. \tag{2.2.5}$$

Denote $x := \{x_k\}_{k=1}^{\infty}$ *and assume that the state space of* (2.2.5) *is*

$$c_0 := \{x : \lim_{k\to\infty} x_k = 0\},$$

with the norm in c_0 *defined by* $\|x\|_{c_0} := \sup_{k\geq 1}|x_k|$.

The solution of (2.2.5) *is* $x(t) = \{e^{-\frac{1}{k}t}x_k(0)\}_{k=1}^{\infty}$.

For any $x(0) \in c_0$ *and for any* $\varepsilon > 0$ *there exist* $N > 0$*:* $\sup_{k\geq N}|x_k(0)| < \varepsilon$*. Since* $|x_k(t)| \leq |x_k(0)|$ *for any* $t \geq 0$*, it holds also that* $\sup_{k\geq N}|x_k(t)| < \varepsilon$ *for any* $t \geq 0$*, and for any* $\varepsilon > 0$.

Moreover, there exists $t^* > 0$*:* $\sup_{1\leq k\leq N-1}|x_k(t)| \leq \varepsilon$ *for* $t \geq t^*$*. Thus, for* $t \geq t^*$ *and for all* $\varepsilon > 0$ *it holds that* $\sup_{k\geq 0}|x_k(t)| < 2\varepsilon$*. Consequently, for any* $x_0 \in c_0$ *the norm of trajectory* $\|x(t)\|_{c_0}$ *converges to* 0 *as long as* $t \to \infty$*, which is exactly global attractivity.*

Since $\|x(t)\|_{c_0} \leq \|x(0)\|_{c_0}$*, for any* $t \geq 0$*, the system* (2.2.5) *is GAS.*

Next we show that (2.2.5) *with a state space* c_0 *is not UGAS. Pick a sequence of states* $\hat{x}_i := (\underbrace{0,\dots,0}_{i-1}, 1, 0, \dots, 0, \dots)$*, for all* $i = 1, \dots, \infty$*. Clearly,* $\|\hat{x}_i\|_{c_0} = 1$ *for all* i*. The solution of* (2.2.5)*, corresponding to initial condition* $\hat{x}_i$*, is equal to* $\phi(t, \hat{x}_i) = e^{-\frac{1}{i}t}\hat{x}_i$.

Assume that $\exists \beta \in \mathcal{KL}$ *so that* $\forall x_0 \in c_0$ *it holds*

$$\|\phi(t, x_0)\|_{c_0} \leq \beta(\|x_0\|_{c_0}, t), \quad t \geq 0. \tag{2.2.6}$$

In particular, this must hold for initial conditions $\hat{x}_i$*, for all* i*. This means that for all* $i = 1, \dots, \infty$ *it holds that*

$$\|\phi(t, \hat{x}_i)\|_{c_0} \leq \beta(1, t), \quad t \geq 0.$$

Due to the properties of $\mathcal{KL}$ *functions, there exist* t^* *s.t.* $\beta(1, t^*) = \frac{1}{2}$*. The above inequality implies that*

$$e^{-\frac{1}{i}t^*} \leq \frac{1}{2} \quad \forall i = 1, \dots, \infty.$$

Clearly this is false. Hence the system under consideration is not UGAS. □

Exercise 2.2.1 (Characterizations of GS)**.** *The system* (2.0.1) *is called practically globally stable (pGS), if there exist* $c > 0$ *and* $\sigma \in \mathcal{K}_\infty$*:* $\forall t \geq 0$ *, for all* $x \in \mathbb{R}^n$

$$|\phi(t, x)| \leq \sigma(|x|) + c. \tag{2.2.7}$$

- *Prove that for* (2.0.1) *it holds that GS = pGS + LS.*
- *Find a restatement of pGS without usage of comparison functions.*
- *Finally, give an* ε-δ *restatement of a GS property.*

Exercise 2.2.2. *In this exercise we continue investigations started in Exercises 2.1.2, 2.1.1, 2.2.1. Our aim is to obtain a more profound picture of relationships between GATT, pGS, GS, nuGS, and continuity properties of the function* $g : x \mapsto \sup_{t\geq 0}|\phi(t, x)|$.

(i) Prove that GATT implies pGS.

(ii) It is easy to see that every pGS system is automatically nuGS. Does the converse hold?

(iii) Prove that if g *is continuous over* $\mathbb{R}^n$*, then* (2.0.1) *is GS. Show by means of a counterexample that converse does not hold.*

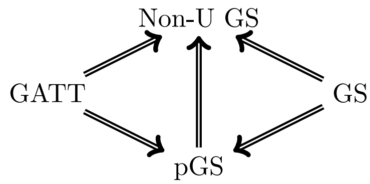

Figure 2.5: Relations between GATT, GS, nuGS and other properties

(iv) Prove that if g is continuous over $\mathbb{R}^n \backslash \{0\}$, then (2.0.1) is pGS. Show by means of a counterexample that converse does not hold.

Exercise 2.2.3. (2.0.1) *has limit property at zero (LIM), if $\exists \gamma \in \mathcal{K}_\infty \cup \{0\}$, such that*

$$\inf_{t \geq 0} |\phi(t,x)| = 0, \quad \forall x \in \mathbb{R}^n.$$

That is, the system (2.0.1) is LIM if every its trajectory approaches the origin arbitrarily close.

Prove that (2.0.1) is LIM+LS if and only if (2.0.1) is GAS.

Exercise 2.2.4. *Obviously, GATT implies LIM. Prove or disprove that LIM implies GATT.*

2.3 Lyapunov functions

Usually it is hard to verify GAS or GS of nonlinear systems (2.0.1) directly. The restatements of GAS property which we have proved in the Section 2.2 are important in theory, but they are not much useful in practice to check GAS of a particular nonlinear system. In this section we develop Lyapunov theory giving a constructive method to prove GAS of nonlinear systems.

Definition 2.3.1. *A continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a strict (GAS) Lyapunov function for the system (2.0.1), if $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{P}$ so that $\forall x \in \mathbb{R}^n$*

$$\psi_1 (|x|) \leq V(x) \leq \psi_2 (|x|) \tag{2.3.1}$$

and for all $x \neq 0$

$$\dot{V}(x) \leq -\alpha (V(x)), \tag{2.3.2}$$

where the Lie derivative of V is defined as follows:

$$\dot{V}(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t,x)) - V(x)). \tag{2.3.3}$$

Definition 2.3.2. *A continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a non-strict (GS) Lyapunov function for the system (2.0.1), if $\exists \psi_1, \psi_2 \in \mathcal{K}_\infty$ so that $\forall x \in \mathbb{R}^n$ (2.3.1) holds and for all $x \neq 0$*

$$\dot{V}(x) \leq 0. \tag{2.3.4}$$

If V is continuously differentiable at any $x \in \mathbb{R}^n$, then the Lie derivative $\dot{V}(x)$ can be computed as

$$\dot{V}(x) = \nabla V \cdot f(x). \tag{2.3.5}$$

If V is Lipschitz continuous functions, then due to Proposition 1.3.1 (Rademacher's theorem) the formula (2.3.5) holds almost everywhere.

Lyapunov functions are in some sense a generalization of an energy of physical systems. In fact, for many mechanical systems the full energy is a Lyapunov function. In Figure 2.6 we show a graph of a

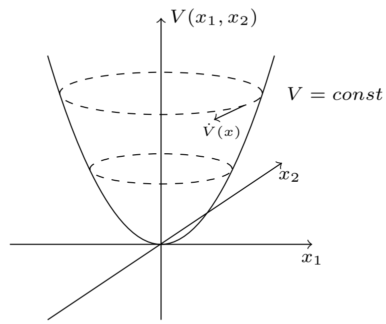

Figure 2.6: Level sets of a typical Lyapunov function V (dashed line)

typical Lyapunov function with several level sets. Since the derivative of a strict Lyapunov function is negative, the state of the system always moves to lower energy levels. In fact, every such trajectory asymptotically approaches the equilibrium, at which the energy of the system equals zero. We will formalize this argument in Theorem 2.3.1. But before that we establish a useful comparison principle which gives a construction of an upper bound for the solution of the 1-dimensional differential inequality:

Lemma 2.3.1. *Let $y : \mathbb{R}_+ \to \mathbb{R}_+$ be an absolutely continuous function defined for all $t \geq 0$ and satisfying differential inequality*

$$\dot{y} \leq -\alpha(y(t)), \quad \textit{for almost all } t, \tag{2.3.6}$$

for some continuous and positive definite function α.

Then there exists a $\beta \in \mathcal{KL}$ so that

$$y(t) \leq \beta(y(0), t) \quad \forall t \geq 0. \tag{2.3.7}$$

Proof. From (2.3.6) it follows that

$$\frac{\dot{y}(t)}{\alpha(y(t))} \leq -1, \quad \text{for almost all } t.$$

Integrating this expression, we have for any $t \geq 0$ that

$$\int_0^t \frac{\dot{y}(s)}{\alpha(y(s))} ds \leq -\int_0^t ds. \tag{2.3.8}$$

Here we allow the integral in the left part of the inequality be equal $-\infty$. Changing the variable as $r := y(s)$, the inequality (2.3.8) can be rewritten as

$$\int_{y(0)}^{y(t)} \frac{dr}{\alpha(r)} \leq -t. \tag{2.3.9}$$

Now define $\eta : \mathbb{R}_+ \to \mathbb{R}$ by

$$\eta(s) := \int_1^s \frac{dr}{\alpha(r)}. \tag{2.3.10}$$

Exploiting this definition we rewrite (2.3.9) as

$$\int_{y(0)}^{y(t)} \frac{dr}{\alpha(r)} = \eta(y(t)) - \eta(y(0)) \leq -t. \tag{2.3.11}$$

Since $\alpha \in \mathcal{P}$, η is a strictly increasing function. Thus, η is invertible and its inverse η^{-1} is also a strictly increasing function, defined over $[\eta(0), \eta(+\infty)] \subset \mathbb{R}$. Easy manipulations with (2.3.11) lead to

$$y(t) \leq \eta^{-1}(\eta(y(0)) - t), \tag{2.3.12}$$

which holds as long as $\eta(y(0))-t \in [\eta(0), \eta(+\infty)]$, i.e. for $t \in [0, \eta(y(0))-\eta(0)]$. At time $t = \eta(y(0))-\eta(0)$ it holds that $y(t) \leq 0$, and thus (since $y(t) \geq 0$ for all $t \geq 0$), $y(t) = 0$. Moreover, the inequality (2.3.6) implies that $y(\tau) = 0$ for all $\tau \geq t$.

Define $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\beta(r,t) := \begin{cases} \eta^{-1}(\eta(r) - t) + re^{-t} & \text{, if } t \in [0, \eta(r) - \eta(0)], \\ re^{-t} & \text{, otherwise.} \end{cases}$$

It is easy to see that $\beta(0,t) = 0 \cdot e^{-t} = 0$ for any $t \geq 0$. Also $\beta(\cdot, t) \in \mathcal{K}_\infty$ for any $t \geq 0$ and $\beta(r, \cdot) \in \mathcal{L}$ for any $r > 0$. Consequently, $\beta \in \mathcal{KL}$ and with this β the inequality (2.3.7) holds. □

The above comparison principle helps to establish the following important theorem:

Theorem 2.3.1. *If there exists a Lyapunov function for (2.0.1), then (2.0.1) is GAS.*

Proof. Pick any $x \neq 0$.

$$\frac{d}{dt} V(x(t)) \leq -\alpha(V(x))$$

From the comparison principle (Lemma 2.3.1) it follows (for $y(t) := V(x(t))$), that $\exists \beta \in \mathcal{KL}$, such that

$$V(x(t)) \leq \beta\left(V(x(0)), t\right).$$

The inequality (2.3.1) implies that for certain $\psi_1, \psi_2 \in \mathcal{K}_\infty$ the following estimate holds

$$\psi_1(|x(t)|) \leq V(x(t)) \leq \beta\left(V(x(0)), t\right) \leq \beta\left(\psi_2(|x(0)|), t\right).$$

This immediately implies

$$|x(t)| \leq \psi_1^{-1} \circ \beta\left(\psi_2(|x(0)|), t\right) := \tilde{\beta}\left(|x(0)|, t\right).$$

Clearly, $\tilde{\beta} \in \mathcal{KL}$ which shows that (2.0.1) is UGAS. □

Exercise 2.3.1. *If there exists a continuous nonstrict Lyapunov function V for (2.0.1), then (2.0.1) is globally stable.*

Exercise 2.3.2. *Let V be a GAS Lyapunov function for (2.0.1). Then for any continuously differentiable $\psi \in \mathcal{K}_\infty$ satisfying $\frac{d\psi}{dr}(r) > 0$ for any $r > 0$ the function $W(x) := \psi(V(x))$ is again a Lyapunov function.*

Next we apply the developed machinery to two well-known systems: to Lorenz' equations and to the nonlinear pendulum

Example 2.3.2. *Consider a famous Lorenz' system*

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{2.3.13}$$

where $\sigma, r, b > 0$.

This system has been proposed by E.N. Lorenz as an extensively simplified model of the atmospheric convection. It is obtained from the complicated system of nonlinear partial differential equations proposed by Rayleigh by considering the dynamics only of three modes of these PDEs, and neglecting all the other

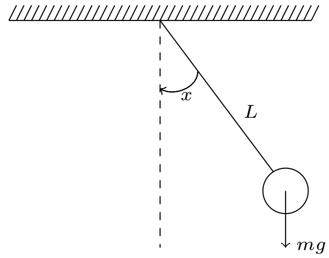

Figure 2.7: Mathematical pendulum

ones. The parameters σ and r are called Prandtl and Rayleigh numbers. System (2.3.13) is famous for being the first discovered system, possessing for certain values of parameters a so-called 'strange attractor'. Our aim in this example is a modest one: we are going to show by means of the Lyapunov criterion, that for $r < 1$ the origin is a GAS equilibrium (and thus a chaotic behavior does not occur).

To this end consider a Lyapunov function

$$V(x,y,z) := \frac{1}{\sigma}x^2 + y^2 + z^2.$$

Lie derivative of V w.r.t. the system (2.3.13) equals

$$\begin{aligned} \frac{d}{dt}V(x,y,z) &= \frac{2}{\sigma}x\dot{x} + 2y\dot{y} + 2z\dot{z} \\ &= 2x(y-x) + 2y(rx - y - xz) + 2z(xy - bz) \\ &= -2x^2 + 2(1+r)xy - 2y^2 - 2bz^2. \end{aligned}$$

For $r < 1$ the last expression is negative for all x,y,z: $x^2+y^2+z^2 \neq 0$. This shows that V is a GAS Lyapunov function for (2.3.13) and thus the origin is a GAS equilibrium.

For $r = 1$ we see that $\frac{d}{dt}V(x,y,z) = -(x-y)^2 - 2bz^2 \leq 0$ for all $x,y,z \in \mathbb{R}$, and $\frac{d}{dt}V(x,y,z) = 0$ iff $x = y$ and $z = 0$. This shows global stability for $r = 1$. Actually, it is possible to show by means of a Barabashin-Krasovskiy-LaSalle principle that for $r = 1$ the system is actually globally asymptotically stable. □

Example 2.3.3. ***Nonlinear pendulum.*** *The equation of motion of a mathematical pendulum (see Figure 2.7) is given by the Newton's law:*

$$\ddot{x} = -\frac{g}{L}\sin x. \tag{2.3.14}$$

Here x is an angle as depicted in Figure 2.7, g is the gravitational acceleration on Earth and L is the length of the string. We assume that $x \in (-\pi, \pi]$.

We are going to analyze stability of the equilibrium $x \equiv 0$ of (2.3.14). First we rewrite (2.3.14) as a system of 2 equations of the first order:

$$\begin{aligned} \dot{x} &= \frac{g}{L}y \\ \dot{y} &= -\sin x \end{aligned} \tag{2.3.15}$$

There are 2 equilibrium points (for $x \in (-\pi, \pi]$) for the nonlinear pendulum: $(x,y) = (0,0)$ and $(x,y) = (\pi, 0)$.

We are going to prove that the equilibrium $(0,0)$ is locally stable, but not locally asymptotically stable.

Let us pick the following Lyapunov function candidate for (2.3.15):

$$V(x,y) := \frac{1}{2}y^2 + \frac{L}{g}(1 - \cos x). \tag{2.3.16}$$

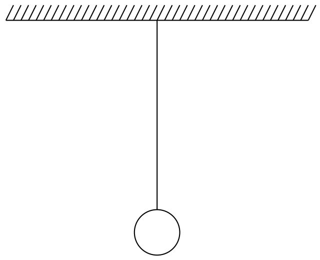

Figure 2.8: The equilibrium $(0, 0)$

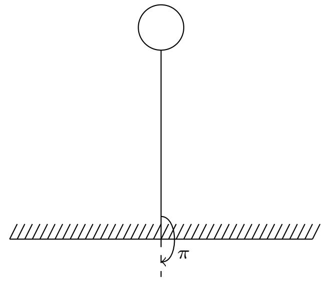

Figure 2.9: The equilibrium $(\pi, 0)$

It is easy to see that $V(0,0) = 0$, and $V(x, y) \neq 0$ unless $x = y = 0$. Lie derivative of V w.r.t (2.3.14) equals:

$$\frac{d}{dt}V(x, y) = y\dot{y} + \frac{L}{g}\dot{x} \cdot \sin x = 0.$$

This means that the energy of a nonlinear pendulum is conserved and thus the system (2.3.15) is locally stable, but not locally asymptotically stable.

Exercise 2.3.3. *Show that the equilibrium $(\pi, 0)$ in the previous example is unstable.*

Exercise 2.3.4. *$V : \mathbb{R}^n \to \mathbb{R}_+$ is called*

- *positive definite if $V(0) = 0$ and $V(x) > 0$ for any $x \neq 0$.*
- *proper (radially unbounded) if $V(x) \to \infty$ as long as $|x| \to \infty$.*

Show that V is positive definite and proper if and only if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$: (2.3.1) holds.

Exercise 2.3.5. *Prove that we obtain the equivalent definition of a GAS-Lyapunov function if we require merely $\dot{V}(x) = \nabla V(x) f(x) < 0$ instead of $\dot{V}(x) = \nabla V(x) f(x) < -\alpha(|x|)$ for some $\alpha \in \mathcal{P}$.*

Comment: This result shows that the trajectories of GAS systems corresponding to the initial values with the same norm, have the uniform speed of convergence. For general infinite-dimensional systems the claim of this exercise is false, even for linear systems, see [36, Ex. 8.2, p. 108].

2.4 Construction of Lyapunov functions for linear GAS systems

There is no general method for construction of Lyapunov functions for asymptotically stable nonlinear systems. In contrast, for linear asymptotically stable systems it is always possible to find a quadratic Lyapunov function and in this section we explain how.

Recall, that a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is called

- positive definite (we write $M > 0$) if $x^T M x > 0$ for all $x \neq 0$.
- negative definite (we write $M < 0$) if $x^T M x < 0$ for all $x \neq 0$.

A matrix $M \in \mathbb{R}^{n \times n}$ is called Hurwitz if $\Re\lambda < 0$ for all $\lambda \in \sigma(M)$, where $\sigma(M)$ is a spectrum of M.

It is well-known that

Proposition 2.4.1. *(2.1.3) is GAS if and only if A is a Hurwitz matrix.*

We need the following lemma, which we give as an exercise for the reader:

Exercise 2.4.1. *Let $L : \mathbb{R}^p \to \mathbb{R}^p$ be a linear operator. If L is a surjection, then it is a bijection.*

The next proposition deals with solutions of the *Sylvester equation*.

Proposition 2.4.2. *Let $M, N \in \mathbb{R}^{p\times p}$ be two Hurwitz matrices. Then for any $Q \in \mathbb{R}^{p\times p}$ there exists the unique solution of the Sylvester equation.*

$$MX + XN = Q \tag{2.4.1}$$

Proof. Consider a linear operator $L : \mathbb{R}^{p\times p} \to \mathbb{R}^{p\times p}$ defined by

$$L(X) := MX + XN.$$

We are going to show that this operator is surjective, which automatically implies by Exercise 2.4.1 that L is a bijection, which in turn shows the claim of the proposition.

To show surjectivity of L we show that for any $Q \in \mathbb{R}^{p\times p}$ the matrix

$$P = -\int_0^\infty e^{Mt} Q e^{Nt} dt$$

solves (2.4.1).

Since M, N are Hurwitz, $\|e^{Mt} Q e^{Nt}\| \leq Ce^{-\lambda t}$ for some $C \geq 1$ and some $\lambda < 0$. Thus,

$$\|P\| \leq C\int_0^\infty e^{-\lambda t} dt < \infty$$

and P is well-defined.

Next we show that P satisfies the equation (2.4.2).

$$\begin{aligned} MP + PN &= -\int_0^\infty Me^{Mt}Qe^{Nt} + e^{Mt}Qe^{Nt}N dt \\ &= -\int_0^\infty \frac{d}{dt}\Big(e^{Mt}Qe^{Nt}\Big) dt \\ &= -\Big(e^{Mt}Qe^{Nt}\Big)\Big|_0^\infty \\ &= Q \end{aligned}$$

Here it was again used that M, N are Hurwitz and thus $\lim_{t\to\infty} \|e^{Mt}\| = \lim_{t\to\infty} \|e^{Nt}\| = 0$. □

Next theorem provies a construction of GAS Lyapunov functions for general GAS linear systems.

Theorem 2.4.3. *A linear system (2.1.3) is GAS if and only if for any $Q > 0$ there exist a unique $P > 0$ which solves the Lyapunov equation*

$$A^T P + PA = -Q. \tag{2.4.2}$$

Moreover, the function

$$V(x) = x^T Px \tag{2.4.3}$$

is a GAS Lyapunov function for a system (2.1.3).

Proof. Pick any $Q > 0$ and let P be the corresponding solution of (2.4.2). We show next that V defined by (2.4.3) is a Lyapunov function for (2.1.3).

Since $P > 0$ it holds that $\lambda_{min}(P)|x|^2 \leq V(x) \leq \lambda_{max}(P)|x|^2$, where $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are the smallest and the largest eigenvalues of P (both positive real numbers). Thus, the condition (2.3.1) is satisfied. Moreover,

$$\begin{aligned}\dot V(x) &= \dot x^T P x + x^T P \dot x \\ &= x A^T P x + x^T P A x \\ &= x^T (A^T P + P A) x \\ &= -x^T Q x \\ &\le -\lambda_{min}(Q)|x|^2, \\ &\le -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} V(x),\end{aligned}$$

for all $x \neq 0$. This shows that V is an exponential GAS Lyapunov function for (2.1.3).

To show the converse claim note that for any $Q \in \mathbb{R}^{n\times n}$ there exists a unique solution of (2.4.2) given by $P = \int_0^\infty (e^{At})^T Q e^{At} dt$. It is easy to see that if $Q > 0$ then also $P > 0$, which finishes the proof. □

Exercise 2.4.2. *Consider a linear operator $L_{AB} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $L_{AB}(X) := AX + XB$. Show that for any $t \in \mathbb{R}$ it holds that $\mathrm{e}^{tL_{AB}}(X) = \mathrm{e}^{tA} X \mathrm{e}^{tB}$.*

Exercise 2.4.3. *Consider an approximation of a nonlinear harmonic oscillator (for small angles x), investigated if Example 2.3.3:*

$$\ddot x + a^2 x = 0$$

for $a > 0$. This approximation is called a harmonic oscillator.

1. *Prove that the solution $x \equiv 0$ is stable but not asymptotically stable.*
2. *Prove a similar statement for the problem*

$$\ddot x + \Omega x = 0,$$

where $\Omega \in \mathbb{R}^{n\times n}$ is a positive definite matrix and $x \in \mathbb{R}^n$.
Hint: *If you are going to use Lyapunov methods for solution of this problem, then it makes sense to rewrite the above problem as a system of the first order with a state (x, y), where $y = \dot x$.*

3. *Find a function $k = k(x, \dot x)$, so that the solution $x \equiv 0$ for*

$$\ddot x + a^2 x = k(x, \dot x)$$

is asymptotically stable.

2.5 Converse Lyapunov theorems

We have introduced a Lyapunov function as a certain generalization of an energy concept. Next we have shown that the unique minimum of a Lyapunov function corresponds to the asymptotically stable equilibrium (or just stable equilibrium provided a Lyapunov function is not strict). Of fundamental importance is the converse question: whether any asymptotically stable system possesses a Lyapunov function, and whether this Lyapunov function can be always chosen to be Lipschitz continuous or even smooth. A great effort has been devoted to this question, and a great number of converse Lyapunov theorems have been proposed. The constructions and proofs of results vary in the difficulty greatly. In particular, the global results are usually harder to prove than local ones and existence of a smooth Lyapunov function is much harder to prove than the existence of a merely continuous one.

We state here the main result which we will use also in the future

Theorem 2.5.1. *Let f be a Lipschitz continuous function on bounded subsets of $\mathbb{R}^n$. If (2.0.1) is GAS, then there exists a smooth GAS Lyapunov function for (2.0.1).*

The corresponding local result is

Theorem 2.5.2. *Let f be a locally Lipschitz continuous function. If (2.0.1) is AS, then there exists a smooth AS Lyapunov function for (2.0.1).*

We will not prove these results. Instead we will prove less strong results but in a more constructive way.

2.6 A simple global converse Lyapunov theorem

We have seen that the existence of a strict Lyapunov function for a system guarantees its GAS. However, currently we do not know whether any GAS system possesses a strict Lyapunov function. In this section we prove a global continuous converse Lyapunov theorem, which guarantees existence of a continuous strict Lyapunov function for any nonlinear system, whose right hand side satisfies basic regularity assumptions. The construction is based upon Sontag's $\mathcal{KL}$-lemma (Proposition 1.2.3).

Next theorem is our first converse Lyapunov theorem for the systems (2.0.1)

Theorem 2.6.1. *If (2.0.1) is UGAS, then there exists a UGAS Lyapunov function for (2.0.1) of the form*

$$V^\gamma(x) := \max_{s\geq 0} e^{\gamma s}\alpha_1(|\phi(s,x)|) \tag{2.6.1}$$

where α_1 comes from (1.2.2) and γ is any number in $(0,1)$.

Proof. Let (2.0.1) be UGAS. Then there exist $\beta \in \mathcal{KL}$, so that for all $x \in \mathbb{R}^n$ and all $t \geq 0$ the estimate (2.2.4) holds. Due to Proposition 1.2.3 there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that (1.2.2) is satisfied.

Consider a function $V : \mathbb{R}^n \to \mathbb{R}_+$, defined by (2.6.1). We are going to prove that V is a UGAS Lyapunov function for (2.0.1). Since $\alpha_1 \in \mathcal{K}_\infty$, the following estimates are justified

$$\begin{aligned} V(x) &= \max_{s\geq 0} e^{\gamma s}\alpha_1(|\phi(s,x)|) \\ &\leq \max_{s\geq 0} e^{\gamma s}\alpha_1(\beta(|x|,s)) \\ &\leq \max_{s\geq 0} e^{\gamma s}\alpha_2(|x|)e^{-s} \\ &= \alpha_2(|x|), \end{aligned}$$

since $\gamma \in (0,1)$.

Due to construction, $V(x) \geq \alpha_1(|x|)$ (to see this just consider $s = 0$ in (2.6.1)).

It remains to show that the Lie derivative of V^γ is negative along the trajectory of (2.0.1). To this end observe that

$$\begin{aligned} V^\gamma(\phi(t,x)) &= \max_{s\geq 0} |e^{\gamma s}\phi(s,\phi(t,x))| \\ &= e^{-\gamma t}\max_{s\geq 0} |e^{\gamma(s+t)}\phi(s+t,x)| \\ &\leq e^{-\gamma t}V^\gamma(x)\,. \end{aligned} \tag{2.6.2}$$

Now we are able to compute the Lie derivative of V^γ.

$$\begin{aligned} \dot V^\gamma(x) &= \overline{\lim_{h\to+0}}\, \frac{1}{h}\big(V^\gamma(\phi(h,x)) - V^\gamma(x)\big) \\ &\leq \overline{\lim_{h\to+0}}\, \frac{1}{h}(e^{-\gamma t}-1)V^\gamma(x) \\ &= -\,\gamma\; V^\gamma(x). \end{aligned}$$

This shows that V^γ is an exponential UGAS Lyapunov function for (2.0.1). $\square$

Next theorem provides an alternative construction of UGAS Lyapunov functions for (2.0.1).

Theorem 2.6.2. *If (2.0.1) is UGAS, then there exist a UGAS Lyapunov function for (2.0.1) of the form*

$$V(x) = \int_0^\infty \alpha_1(|\phi(t,x)|)dt, \tag{2.6.3}$$

where α_1 comes from (1.2.2).

Proof. Let the system (2.0.1) be UGAS. Then there exist $\beta \in \mathcal{KL}$, so that for all $x \in \mathbb{R}^n$ and all $t \geq 0$ the estimate (2.2.4) holds. Due to Lemma 1.2.3 there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that (1.2.2) is satisfied. Now consider a function $V : \mathbb{R}^n \to \mathbb{R}_+$, defined by (2.6.3). We are going to prove that V is a UGAS Lyapunov function for (2.0.1). First of all, since $\alpha_1 \in \mathcal{K}_\infty$, the following estimates are justified

$$\begin{aligned} V(x) &= \int_0^\infty \alpha_1(|\phi(t,x)|)dt \\ &\leq \int_0^\infty \alpha_1(\beta(|x|,t))dt \\ &\leq \int_0^\infty \alpha_2(|x|)e^{-t}dt \\ &= \alpha_2(|x|). \end{aligned}$$

Clearly, $V(x) \geq 0$ for all $x \in \mathbb{R}^n$. Let $V(x) = 0$. This implies that $\phi(t,x) = 0$ for all $t \geq 0$, and thus $x = 0$. Thus, $V(x) > 0$ for all $x \neq 0$ and V is positive definite.

The Lie derivative of V at any $x \in \mathbb{R}^n$ is given by:

$$\begin{aligned} \dot{V}(x) &= \overline{\lim_{h\to+0}} \frac{1}{h}\big(V(\phi(h,x)) - V(x)\big) \\ &= \overline{\lim_{h\to+0}} \frac{1}{h}\left(\int_0^\infty \alpha_1(|\phi(t,\phi(h,x))|)dt - \int_0^\infty \alpha_1(|\phi(t,x)|)dt\right) \\ &= \overline{\lim_{h\to+0}} \frac{1}{h}\left(\int_0^\infty \alpha_1(|\phi(t+h,x)|)dt - \int_0^\infty \alpha_1(|\phi(t,x)|)dt\right) \\ &= \overline{\lim_{h\to+0}} \frac{1}{h}\left(\int_h^\infty \alpha_1(|\phi(t,x)|)dt - \int_0^\infty \alpha_1(|\phi(t,x)|)dt\right) \\ &= -\overline{\lim_{h\to+0}} \frac{1}{h}\left(\int_0^h \alpha_1(|\phi(t,x)|)dt\right) \\ &= -\alpha_1(|x|) \end{aligned}$$

It remains to prove existence of a lower bound for V. Define for any $r \geq 0$

$$\psi(r) := \inf_{x\in\mathbb{R}^n : |x| \geq r} \int_0^\infty \alpha_1(|\phi(t,x)|)dt. \tag{2.6.4}$$

By construction, $\psi(\cdot)$ is a nondecreasing function and also $V(x) \geq \psi(|x|)$ for any $x \in \mathbb{R}^n$. Since $\phi(t,0) = 0$ for all $t \geq 0$, we have $\psi(0) = 0$.

We are going to prove that $\psi(r) > 0$ for $r > 0$. To this end first we are going to prove the claim that the infimum in the integral in (2.6.4) is attained on the set $\{x \in \mathbb{R}^n : |x| = r\}$. Indeed, pick any $x \in \mathbb{R}^n$: $|x| > r$. Since (2.0.1) is UGAS, $\phi(t,x) \to 0$ as soon as $t \to \infty$. Thus there exists some $t' > 0$: $|\phi(t',x)| = r$. According to semigroup property for any $t \geq t'$ it holds that $\phi(t,x) = \phi(t-t', \phi(t',x))$. In turn, this equality leads to

$$\begin{aligned}\int_0^\infty \alpha_1(|\phi(t,x)|)dt &= \int_0^{t'} \alpha_1(|\phi(t,x)|)dt + \int_{t'}^\infty \alpha_1(|\phi(t,x)|)dt \\ &= \int_0^{t'} \alpha_1(|\phi(t,x)|)dt + \int_{t'}^\infty \alpha_1(|\phi(t-t',\phi(t',x))|)dt \\ &= \int_0^{t'} \alpha_1(|\phi(t,x)|)dt + \int_0^\infty \alpha_1(|\phi(s,y)|)ds,\end{aligned}$$

where $y = \phi(t', x)$ with $|y| = r$. These equalities show that for any $x \in \mathbb{R}^n$: $|x| > r$ there exist $y \in \mathbb{R}^n$: $|y| = r$ so that

$$\int_0^\infty \alpha_1(|\phi(t,x)|)dt > \int_0^\infty \alpha_1(|\phi(t,y)|)dt.$$

This proves the claim and thus ψ can be alternatively defined as

$$\psi(r) = \inf_{x\in\mathbb{R}^n:|x|=r} \int_0^\infty \alpha_1(|\phi(t,x)|)dt. \tag{2.6.5}$$

Since $\{x \in \mathbb{R}^n : |x| = r\}$ is compact, and since for any $x : |x| = r$ it holds that $\int_0^\infty \alpha_1(|\phi(t,x)|)dt > 0$, it follows that $\psi(r) > 0$ for any $r > 0$.

Define $\psi_1(r) := (1-e^{-r})\psi(r)$ for any $r \geq 0$. Clearly, $\psi_1(r) \leq \psi(r)$ and $\psi_1 \in \mathcal{K}_\infty$. Thus, $V(x) \geq \psi_1(|x|)$ and consequently V is a UGAS Lyapunov function for (2.0.1). □

2.7 Summary: advantages and limitations of the classical stability theory

Our investigations of GAS of (2.0.1) we summarize in

Theorem 2.7.1. *The following properties are equivalent:*

1. *(2.0.1) is GAS*
2. *(2.0.1) is UGAS*
3. *(2.0.1) is UGATT*
4. *(2.0.1) is LIM + LS*
5. *There exists a smooth GAS Lyapunov function for (2.0.1)*

Proof. The claim follows from Theorems 2.2.1, 2.5.1 and Excercise 2.2.3. □

As we see the GAS property has a number of ultimately useful restatements, which make it important in practice. However, in spite of these advantages classical stability theory is not sufficient for systems with external inputs (or disturbances):

$$\dot{x} = f(x,u), \ t > 0.$$

Even if this system is globally asymptotically stable for $u \equiv 0$, this does not necessarily mean, that the system will have 'good' properties for a nonzero input, even if it is chosen arbitrarily small. This issue makes a lot of troubles for engineers, since small disturbances always occur in real-world systems.

Another problem of a classical stability theory is a failure to consider interconnections of dynamical systems. Indeed, let the state of a system be divided into two parts $x = (x_1, x_2)$. We can rewrite the dynamics of the system as follows:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2). \end{cases}$$

A reasonable question arises: what we can say about stability of the whole system if we know the stability properties of its subsystems? It turns out that the classical stability theory does not have tools to answer this question properly, since the subsystems of the interconnected system are no more dynamical systems, but control systems, possessing a state of the other subsystem as an input.

Last but not least, Lyapunov stability theory is not the only stability concept used in the control theory. Another approach is a so-called input-output stability. These approaches were complementary to each other, and both have a rich theory. A fundamental question would be to unify these two theories.

All these problems have been successfully solved within input-to-state stability (ISS) theory.

2.8 Concluding remarks

The stability theory of dynamical systems has been created since the groundbreaking dissertation "Obshchaya zadacha ob ustoichivosti dvizhenia" ('General problem of stability of motion') by the Russian mathematician A. M. Lyapunov, published in 1892 [58] (Engl. translation: [59]).

We developed stability theory of dynamical systems in this section in a way to provide a firm basis for the further development of input-to-state stability theory. Therefore many interesting topics in dynamical systems theory have been omitted. For a more broad picture on results and methods of dynamical systems theory for systems without disturbances you may consult many excellent books, e.g. [75, 9, 61].

Chapter 3

ISS and Lyapunov methods

In this section we proceed to the study of systems with inputs of the form

$$\begin{cases} \dot{x} = f(x,u), \ t > 0 \\ x(0) = x_0. \end{cases} \tag{3.0.1}$$

where $x(t) \in \mathbb{R}^n$, $u \in \mathcal{U} = L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and we continue to exploit Assumption 1.1.1.

3.1 Basic definitions and results

The next notion will be central in these notes:

Definition 3.1.1. *System (3.0.1) is called input-to-state stable (ISS), if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that $\forall x \in \mathbb{R}^n$, $\forall u \in \mathcal{U}$ and $\forall t \geq 0$ the following holds*

$$|\phi(t,x,u)| \leq \beta(|x|,t) + \gamma(\|u\|_\infty). \tag{3.1.1}$$

The function γ is called *gain* and describes the influence of the input on the system. The β describes the transient behavior of the system.

Definition 3.1.2. *We call the system (3.0.1) globally asymptotically stable at zero (0-GAS), if the system (3.0.1) with $u \equiv 0$ is GAS. The properties 0-GS, 0-GATT, 0-LS etc are defined analogously.*

In Figure 3.1 a trajectory of a typical ISS system is depicted. Substituting $u \equiv 0$ into the definition of ISS, we see immediately that any ISS system is 0-GAS. On the other hand, taking the upper limit $t \to \infty$, we see that the trajectory of any ISS system satisfies for any $x \in \mathbb{R}^n$ and any $u \in \mathcal{U}$ the property

$$\limsup_{t\to\infty} |\phi(t,x,u)| \leq \gamma(\|u\|_\infty), \tag{3.1.2}$$

called the *asymptotic gain property (AG)*. A trajectory of a typical AG system is described in Figure 3.2. Every trajectory of an AG system converges to the neighborhood of an origin with a radius $\gamma(\|u\|_\infty)$.

Remark 3.1.1. *We have defined ISS property in the 'sum form'. Alternatively, ISS can be defined in the max form: (3.0.1) is ISS if and only if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that $\forall x \in \mathbb{R}^n$, $\forall u \in \mathcal{U}$ and $\forall t \geq 0$ the following holds*

$$|\phi(t,x,u)| \leq \max\{\beta(|x|,t), \gamma(\|u\|_\infty)\}. \tag{3.1.3}$$

This is due to the inequality $\max\{r,s\} \leq r+s \leq 2\max\{r,s\}$, which holds for any $r,s \in \mathbb{R}_+$. Certainly β and γ may be different in max and sum formulations.

ISS systems have many interesting properties. To begin with let us prove the following:

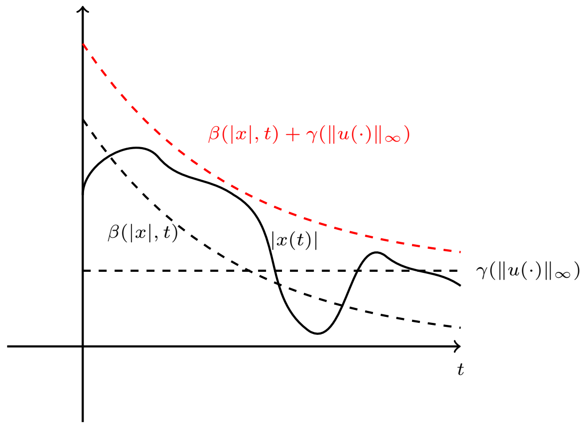

Figure 3.1: Typical trajectory of an ISS system

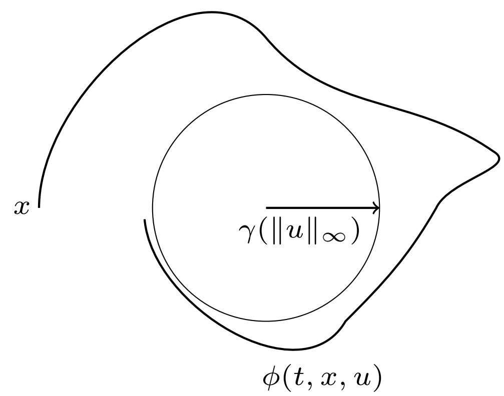

Figure 3.2: Typical trajectory of a system, possessing an asymptotic gain.

Proposition 3.1.1. *Let (3.0.1) be ISS. If* $\lim_{t\to\infty}|u(t)|=0$*, then* $\lim_{t\to\infty}|\phi(t,x,u)|=0$ *for any* $x\in\mathbb{R}^n$*.*

Proof. Let (3.0.1) be ISS and let $\lim_{t\to\infty}|u(t)|=0$. Pick any $x\in\mathbb{R}^n$. For every $k\in\mathbb{N}$ there exists time τ_k (depending on $|x|$) so that $\sup_{t\geq\tau_k}|u(t)|\leq\gamma^{-1}(\frac{|x|}{2^{k+1}})$. Without loss of generality we can assume that this sequence is monotonically increasing to infinity. Denote $T_1:=\tau_1$.

Due to the semigroup property (1.1.6) it holds that

$$\phi(t+T_1,x,u)=\phi(t,\phi(T_1,x,u),u(\cdot+T_1)).$$

Hence

$$|\phi(t+T_1,x,u)|\leq\beta(|\phi(T_1,x,u)|,t)+\gamma(\|u(\cdot+T_1)\|_\infty).$$

Pick any T_2 in a way that $\beta(|\phi(T_1,x,u)|,T_2)\leq\frac{|x|}{4}$ and $T_2+T_1\geq\tau_2$.

Since $\gamma(\|u(\cdot+T_1)\|_\infty)\leq\frac{|x|}{4}$ we obtain that

$$|\phi(T_2+T_1,x,u)|\leq\frac{|x|}{2}$$

and

$$|\phi(t+T_2+T_1,x,u)|\leq\beta\big(\tfrac{|x|}{2},t\big)+\|u(\cdot+T_2+T_1)\|_\infty.$$

Now pick a sequence T_k, satisfying the following properties: $\beta(\frac{|x|}{2^{k-2}}, T_k) \leq \frac{|x|}{2^k}$, and $T_k+T_{k-1}+\ldots+T_1 \geq \tau_k$ for $k = 3, 4, \ldots, \infty$.

By induction, we obtain that for any $k \geq 3$

$$|\phi(t + T_k + \ldots + T_1, x, u)| \leq \frac{|x|}{2^{k-1}}, \quad t \geq 0.$$

Taking limit $k \to \infty$ shows the claim. □

For nonlinear systems ISS is much stronger that 0-GAS property. In contrast to that for linear systems both these properties are equivalent, which we show next:

Proposition 3.1.2. *A linear system*

$$\dot{x} = Ax + Bu \tag{3.1.4}$$

is ISS ⇔ it is exponentially ISS ⇔ it is 0-GAS.

Proof. First we prove that 0-GAS implies exponential ISS.

The solution of a system (3.1.4) subject to initial condition $x(0) = x$ and input $u \in \mathcal{U}$ is given by

$$\phi(t, x, u) = e^{At}x + \int_0^t e^{A(t-s)}Bu(s)ds. \tag{3.1.5}$$

Since (3.1.4) is 0-GAS, A is a Hurwitz matrix and $\|e^{At}\| \leq Me^{\lambda t}$ for some $M > 0$, $\lambda < 0$ and for all $t \geq 0$. This leads to

$$\begin{aligned} |\phi(t, x, u)| &\leq |e^{At}x| + \int_0^t \|e^{A(t-s)}\| |Bu(s)| ds \\ &\leq Me^{\lambda t}|x| + M \int_0^t e^{\lambda(t-s)} ds \|B\| \|u\|_\infty. \end{aligned}$$

The integral in right hand side can be easily computed for any $t > 0$:

$$\int_0^t e^{\lambda(t-s)} ds = \frac{1}{|\lambda|}(1 - e^{\lambda t}) \leq \frac{1}{|\lambda|}.$$

Substitution of this estimate into the previous calculations shows that (3.1.4) is exponentially ISS and its trajectory satisfies

$$|\phi(t, x, u)| \leq Me^{\lambda t}|x| + \frac{M}{|\lambda|}\|B\| \|u\|_\infty.$$

Since exponential ISS implies ISS, which in turn implies 0-GAS, the claim of the proposition is proved. □

For nonlinear systems 0-GAS $\neq$ ISS, which can be easily seen from the following example:

$$\dot{x} = -x + (1 + x^2)u, \quad x(t), u(t) \in \mathbb{R}.$$

This system is 0-GAS, but for $u \equiv 1$ the trajectory of this system exhibits blow-up for any initial condition (show this!).

Exercise 3.1.1. *Prove that (3.0.1) is ISS if and only if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that $\forall x \in \mathbb{R}^n$, $\forall u \in \mathcal{U}$ and $\forall t \geq 0$ the following holds*

$$|\phi(t, x, u)| \leq \beta(|x|, t) + \gamma(\sup_{0 \leq s \leq t} |u(s)|). \tag{3.1.6}$$

Exercise 3.1.2. *Prove that* (3.0.1) *is ISS if and only if there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such that* (3.1.1) *holds* $\forall x \in \mathbb{R}^n$, $\forall u \in \mathcal{U}$ *and* $\forall t \in [0, t_{max})$, *where* $t_{max} = t_{max}(x,u)$ *is the maximal time of existence of solutions of* (3.0.1).

Exercise 3.1.3. *Let* $\mathcal{KKL}$ *be a class of functions* $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, *which satisfy* $\omega(\cdot, r, t) \in \mathcal{K}$ *for any* r, t, $\omega(r, \cdot, t) \in \mathcal{K}$ *for any* r, t, $\omega(r_1, r_2, \cdot) \in \mathcal{L}$ *for any* $r_1 \neq 0, r_2 \neq 0$.

Prove, that if there exists $\omega \in \mathcal{KKL}$ *so that for all* $x \in \mathbb{R}^n$, $u \in \mathcal{U}$ *and* $t \geq 0$ *it holds that*

$$|\phi(t,x,u)| \leq \omega(|x|, \|u\|_\infty, t),$$

then (3.0.1) *is ISS.*

3.2 ISS Lyapunov functions

In this section we show that Lyapunov methodology, which we exploited earlier to study 0-GAS of dynamical systems, can be successfully applied to study ISS of nonlinear control systems.

Definition 3.2.1. *A Lipschitz continuous function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *is called an ISS Lyapunov function, if there exist* $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{K}_\infty$ *and* $\chi \in \mathcal{K}$ *such that*

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^n \tag{3.2.1}$$

and for any $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ *the following implication holds:*

$$V(x) \geq \chi(|u|) \quad \Rightarrow \quad \nabla V(x) \cdot f(x,u) \leq -\alpha(V(x)). \tag{3.2.2}$$

The function χ *is often called a Lyapunov gain.*

If we specify Definition 3.2.1 to systems without inputs (by setting $u \equiv 0$), we see that an ISS Lyapunov function is automatically a 0-GAS Lyapunov function. If $u \not\equiv 0$, then existence of an ISS Lyapunov function does not guarantee a convergence of all trajectories to zero. However, according to the implication (3.2.2), if $V(x) \geq \chi(|u|)$ holds, the Lie derivative of V is negative and thus the trajectory converges to the region $\{x \in \mathbb{R}^n : V(x) \leq \chi(\|u\|_\infty)\}$. We will render this argument more precisely in Theorem 3.2.2.

In Definition 3.2.1 we defined an *ISS Lyapunov function in an implication form*. Next we show that another equivalent definition *in a dissipative form* is possible.

Proposition 3.2.1. *A Lipschitz continuous function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *is an ISS Lyapunov function if and only if there exist* $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{K}_\infty$ *and* $\xi \in \mathcal{K}$ *such that* (3.2.1) *holds and for any* $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ *the following dissipation inequality holds:*

$$\dot{V}(x) = \nabla V(x) \cdot f(x,u) \leq -\alpha(V(x)) + \xi(|u|). \tag{3.2.3}$$

Proof. Let V be a Lyapunov function in a dissipative form so that (3.2.3) is satisfied. Whenever $|u| \leq \xi^{-1}(\frac{1}{2}\alpha(V(x)))$ holds, which is the same as $V(x) \geq \alpha^{-1}(2\xi(|u|))$, the inequality

$$\nabla V(x) \cdot f(x,u) \leq -\frac{1}{2}\alpha(V(x))$$

is satisfied, which means that V is an ISS Lyapunov function in an implicative form with a Lyapunov gain $\chi(\cdot) := \alpha^{-1}(2\xi(\cdot))$.

Now let V be a Lyapunov function in an implicative form and thus (3.2.2) holds. Define $\bar{\xi}(r) := \max\{0, \xi^*(r)\}$, where

$$\xi^*(r) := \max\{\nabla V(x) \cdot f(x,v) + \alpha(V(x)) : |v| \leq r,\ V(x) \leq \chi(r)\}.$$

The maximum in the definition of ξ^* exists since we are maximizing over a compact set. Moreover, $\bar{\xi}$ is continuous, nondecreasing and $\bar{\xi}(0) = 0$. Thus, it can be always majorized by a $\mathcal{K}_\infty$-function ξ in a way that $\xi(r) \geq \bar{\xi}(r)$ for all $r \in \mathbb{R}_+$.

Now for all $x \in \mathbb{R}^n, u \in \mathbb{R}^m$: $V(x) \leq \chi(|u|)$ we have

$$\nabla V(x) \cdot f(x,u) + \alpha(V(x)) \leq \xi(|u|). \tag{3.2.4}$$

with α from (3.2.2) and with the constructed ξ.

Combining this inequality with (3.2.2) we see that (3.2.3) holds for all x, u which shows that V is an ISS Lyapunov function in a dissipative form. □

The following theorem is a basis for a great part of applications of ISS theory

Theorem 3.2.2. *Existence of an ISS Lyapunov function for* (3.0.1) *implies ISS of* (3.0.1).

Proof. Pick any $u \in \mathcal{U}$ and fix it. Consider a set $S = \{x \in \mathbb{R}^n : V(x) \leq \chi(\|u\|_\infty)\}$. We are going to prove that S is invariant w.r.t. flow, that is $x \in S$ implies $\phi(t,x,u) \in S$ for all $t \geq 0$.

Assume that for certain $x \in S$ there exists $t_1 < \infty$ so that $\phi(t_1,x,u) \notin S$. Denote for short $x(t) := \phi(t,x,u)$. Define $t^* = \inf\{t \in \mathbb{R}_+ : \phi(t,x,u) \notin S\}$. We have

$$V\left(x(t^*)\right) \geq \chi(\|u\|_\infty).$$

which due to (3.2.2) leads to

$$\dot{V}\left(x(t^*)\right) \leq -\alpha\left(V(x(t^*))\right).$$

Consequently $V(x(t)) > V(x(t^*))$ for some $t < t^*$, which contradicts to the minimality of t^*.

Overall,

$$\psi_1\left(|\phi(t,x,u)|\right) \leq V(\phi(t,x,u)) \leq \chi(\|u\|_\infty), \quad t \geq 0,$$

which implies that

$$|\phi(t,x,u)| \leq \psi_1^{-1} \circ \chi(\|u\|_\infty). \tag{3.2.5}$$

Now let $x \notin S$, i.e. $V(x) > \chi(\|u\|_\infty)$. This implies that

$$\frac{d}{dt}V(x(t)) \leq -\alpha(V(x)).$$

Due to comparison principle (Lemma 2.3.1) the above inequality necessarily implies existence of $\beta \in \mathcal{KL}$ so that

$$V(x(t)) \leq \beta\left(V(x(0)), t\right),$$

for all t so that $V(x(t)) > \chi(\|u\|_\infty)$. Thus,

$$\psi_1(|x(t)|) \leq V(x(t)) \leq \beta\left(\psi_2(|x(0)|), t\right), \quad t \leq \tau, \tag{3.2.6}$$

where $\tau = \inf\{t : |x(t)| \leq \chi(\|u\|_\infty)\}$.

Since for $t \geq \tau$ the estimate (3.2.5) is valid, we conclude from (3.2.5) and (3.2.6) that

$$|\phi(t,x,u)| \leq \tilde{\beta}\left(|x|, t\right) + \gamma(\|u\|_\infty) \quad \forall t > 0,$$

where $\tilde{\beta}(r,t) := \psi_1^{-1}\big(\beta(\psi_2(r),t)\big)$ for any $r, t \geq 0$. This proves, that system (3.0.1) is ISS. □

In view of converse Lyapunov theorems for dynamical systems it is natural to ask for their ISS counterparts. The converse results have been proved in [70] on the basis of the paper [55]. In these notes we state this foundational result without the proof:

Theorem 3.2.3. (3.0.1) *is ISS if and only if there exist a smooth ISS Lyapunov function for* (3.0.1).

Proof. For a proof please consult [70, 55]. □

Later in Section 3.3 we will prove a converse Lyapunov result for local ISS property (which is much easier to prove than the global result).

Exercise 3.2.1. *Show that replacing the inequality (3.2.2) in Definition 3.2.1 with the one below*

$$|x| \geq \chi(|u|) \quad \Rightarrow \quad \nabla V(x) \cdot f(x,u) \leq -\alpha(|x|), \tag{3.2.7}$$

we obtain an equivalent definition of an ISS Lyapunov function.

Exercise 3.2.2. *Prove that replacing the requirement $\alpha \in \mathcal{K}_\infty$ in Definition 3.2.1 with $\alpha \in \mathcal{P}$ we obtain an equivalent definition of an ISS-Lyapunov function.*

Exercise 3.2.3. *Let us replace the inequality (3.2.2) in Definition 3.2.1 with the following one*

$$V(x) \geq \chi(|u|) \quad \Rightarrow \quad \nabla V(x) \cdot f(x,u) < 0. \tag{3.2.8}$$

Will the existence of such a function V imply ISS?

In the next section we show how the developed machinery helps to stabilize certain classes of nonlinear control systems in a robust way. But before that we demonstrate the applicability of the Lyapunov method on an academic example.

Example 3.2.4. *We are going to prove that*

$$\dot{x} = -x + u \ln(|x|+1), \;\; x(t), u(t) \in \mathbb{R}. \tag{3.2.9}$$

is ISS. Consider an ISS Lyapunov function candidate $V(x) = x^2$. Let us compute the Lie derivative of V:

$$\begin{aligned} \dot{V}(x) = \nabla V(x) \cdot f(x,u) &= 2x\left(-x + u\log(|x|+1)\right) \\ &\leq -2x^2 + 2|x||u|\log(|x|+1). \end{aligned}$$

Define

$$\chi^{-1}(r) := \begin{cases} \frac{r}{2\log(r+1)} - \frac{1}{2} & , \text{ if } r > 0, \\ 0 & , \text{ if } r = 0. \end{cases}$$

It is an easy exercise to show that χ^{-1} is continuous and strictly increasing to infinity. In other words, $\chi^{-1} \in \mathcal{K}_\infty$ and thus also $\chi \in \mathcal{K}_\infty$.

Then $\chi(|u|) \leq |x|$ (which is the same as $|u| \leq \chi^{-1}(|x|) = \frac{|x|}{2\log(|x|+1)} - \frac{1}{2}$) implies

$$\nabla V(x) \cdot f(x,u) \leq -|x|^2 - |x|\log(|x|+1) =: -\alpha(|x|).$$

This shows that (3.2.9) is ISS (we are using here Exercise 3.2.1).

Exercise 3.2.4. *Let $x_i(t) \in \mathbb{R}$, $i = 1,2$ and $u(t) \in \mathbb{R}$. Prove that the following system is ISS (Taken from [38]).*

$$\dot{x}_1 = -x_1 + x_1^3 x_2 \tag{3.2.10}$$

$$\dot{x}_2 = -x_2 - x_1^6 x_2 + u \tag{3.2.11}$$

Exercise 3.2.5. *Prove that a linear system (3.1.4) is ISS if and only if there exist a quadratic ISS-Lyapunov function $V(x) = x^T P x$ for some $P \in \mathbb{R}^{n\times n}$ so that $P = P^T > 0$.*

Hint: since (3.1.4) is 0-GAS, $V(x) = x^T P x$ is a 0-GAS Lyapunov function, if $P = P^T > 0$ is chosen s.t.

$$A^T P + PA = -I$$

Prove that V is an ISS Lyapunov function for properly chosen Lyapunov gains.

3.3 Converse Lyapunov theorem for LISS property

We have seen that 0-GAS systems are seldom ISS. In this section we introduce a local input-to-state stability (LISS) notion and prove that under reasonable requirements on the right-hand side local asymptotical stability at zero (0-AS) is equivalent to LISS.

Definition 3.3.1. *System (3.0.1) is called locally input-to-state stable (LISS), if there exist* $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ *and* $r > 0$ *such that the inequality (3.1.1) holds* $\forall x \in B_r$, $\forall u \in \mathcal{U}$: $\|u\|_\infty \le r$ *and* $\forall t \ge 0$.

The corresponding notion of LISS Lyapunov function is defined as:

Definition 3.3.2. *A continuous function* $V : D \to \mathbb{R}_+$, $0 \in int(D) \subset \mathbb{R}^n$ *is called a LISS Lyapunov function, if there exist* $r > 0$, $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{K}_\infty$ *and* $\sigma \in \mathcal{K}$ *such that* $B_r \subset D$ *and*

$$\psi_1(|x|) \le V(x) \le \psi_2(|x|), \quad \forall x \in B_r \tag{3.3.1}$$

and Lie derivative of V *along the trajectories of the system (3.0.1) satisfies (3.2.3) for all* $x \in B_r$ *and* $u \in \mathcal{U}$: $\|u\|_\infty \le r$.

Analogously to Theorem 3.3.1 one can obtain the following result:

Theorem 3.3.1. *If* $\exists$ *LISS-LF for (3.0.1)* $\quad \Rightarrow \quad$ *(3.0.1) is LISS.*

Next we prove the converse statement, namely that LISS implies existence of a LISS Lyapunov function. We rely upon the converse Lyapunov theorem for undisturbed systems (3.0.1).

Lemma 3.3.1. *Assumption 1.1.1, item (ii) implies that there exist* $\sigma \in \mathcal{K}$ *and* $\rho > 0$ *so that for all* $v \in U$: $|v\| \le \rho$ *and all* $x \in \mathbb{R}^n$: $|x| \le \rho$ *we have*

$$|f(x,v) - f(x,0)| \le \sigma(|v|). \tag{3.3.2}$$

Proof. One of possible σ is the following one:

$$\sigma(r) := \sup_{v \in \mathbb{R}^m : |v| \le r} \; \sup_{x \in \mathbb{R}^n : |x| \le \rho} |f(x,v) - f(x,0)| + r$$

The supremum exists due to compactness of closed balls in $\mathbb{R}^m$. □

Next result (which is closely related to the known fact about robustness of the 0-UAS property [27, Corollary 4.2.3]) shows, that a Lipschitz continuous 0-UAS Lyapunov function for (3.0.1) is, under a certain assumption on the nonlinearity f, also a LISS Lyapunov function for (3.0.1).

Proposition 3.3.2. *Let Assumption 1.1.1 hold and let Let* V *be a Lipschitz continuous 0-UAS Lyapunov function for (3.0.1). Then* V *is also a LISS Lyapunov function for (3.0.1).*

Proof. Let $V : D \to \mathbb{R}_+$, $D \subset \mathbb{R}^n$, with $0 \in \text{int}(D)$ be a Lipschitz continuous (0-UAS) Lyapunov function for (3.0.1), which satisfies (2.3.2) for $x \in \mathbb{R}^n$: $|x| \le r$. Due to continuous dependence of ϕ on inputs and initial values, there exist $r_2 > 0$ and $t^* > 0$ so that for all $x \in \mathbb{R}^n : |x| \le r$ and all $u \in \mathcal{U}$: $\|u\|_\infty \le r_2$ the solution $\phi(s,x,u)$ exists for $s \in [0,t^*]$ and $|\phi(s,x,u)| \le 2r$ for all $s \in [0,t^*]$. Let us assume without restriction of generality, that $r_2 < \rho$ and $2r < \rho$.

We are going to prove that V is a LISS Lyapunov function for (3.0.1). To this end we are going to derive a dissipative estimate for $\dot{V}_u(x)$ for all x, u: $|x| \le r$ and $\|u\|_\infty \le r_2$. We have:

$$\begin{aligned}
\dot{V}_u(x) &= \overline{\lim_{t \to +0}} \frac{1}{t}\Big(V\big(\phi(t,x,u)\big) - V(x)\Big) \\
&= \overline{\lim_{t \to +0}} \frac{1}{t}\Big(V\big(\phi(t,x,0)\big) - V(x) + V\big(\phi(t,x,u)\big) - V\big(\phi(t,x,0)\big)\Big) \\
&= \dot{V}_0(x) + \overline{\lim_{t \to +0}} \frac{1}{t}\Big(V\big(\phi(t,x,u)\big) - V\big(\phi(t,x,0)\big)\Big)
\end{aligned}$$

Since V is a 0-UAS Lyapunov function for (3.0.1), due to (2.3.2) it holds for a certain $\alpha \in \mathcal{K}_\infty$ that

$$\dot{V}_u(x) \quad \leq \quad -\alpha(|x|) + \overline{\lim_{t\to+0}} \frac{1}{t}\Big|V\big(\phi(t,x,u)\big) - V\big(\phi(t,x,0)\big)\Big|.$$

Since $\phi(t,x,u) \in D$ for all x, u: $|x| \leq r$ and $\|u\|_\infty \leq r_2$, and since V is Lipschitz continuous on D, there exists $L > 0$ so that

$$\dot{V}_u(x) \leq -\alpha(|x|) + L \overline{\lim_{t\to+0}} \frac{1}{t}|\phi(t,x,u) - \phi(t,x,0)|. \tag{3.3.3}$$

Now we are going to obtain an estimate for $|\phi(t,x,u)-\phi(t,x,0)|$ for $t \in [0,t^*]$. The variation of constants formula implies:

$$\begin{aligned}
|\phi(t,x,u) - \phi(t,x,0)| \quad &= \quad \Big|\int_0^t e^{(t-s)A}\Big(f(\phi(s,x,u),u(s)) - f(\phi(s,x,0),0)\Big)ds\Big| \\
&\leq \quad \int_0^t \Big|e^{(t-s)A}\Big|\Big|f(\phi(s,x,u),u(s)) - f(\phi(s,x,0),0)\Big|ds.
\end{aligned}$$

Since $|e^{(t-s)A}| \leq M$ for all $s \in [0,t]$ and certain $M > 0$, we proceed to

$$\begin{aligned}
|\phi(t,x,u) - \phi(t,x,0)| \quad \leq \quad &\int_0^t M\Big(|f(\phi(s,x,u),u(s)) - f(\phi(s,x,0),u(s))| \\
&+|f(\phi(s,x,0),u(s)) - f(\phi(s,x,0),0)|\Big)ds.
\end{aligned}$$

Recalling that $|\phi(t,x,u)| \leq 2r$ for all $t \in [0,t^*]$, due to Lipschitz continuity of f w.r.t. the first argument, there exist $L_2 > 0$:

$$\begin{aligned}
|\phi(t,x,u) - \phi(t,x,0)| \quad &\leq \quad ML_2\int_0^t \big|\phi(s,x,u) - \phi(s,x,0)\big|ds + Mt\sigma\big(\sup_{0\leq s\leq t}|u(s)|\big) \\
&\leq \quad ML_2 t\sup_{0\leq s\leq t}|\phi(s,x,u) - \phi(s,x,0)| + Mt\sigma\big(\operatorname*{ess\,sup}_{0\leq s\leq t}|u(s)|\big).
\end{aligned}$$

The right hand side of the above inequality is nondecreasing in t and consequently it holds that

$$\sup_{0\leq s\leq t}|\phi(s,x,u) - \phi(s,x,0)| \quad \leq \quad ML_2 t\sup_{0\leq s\leq t}|\phi(s,x,u) - \phi(s,x,0)| + Mt\sigma\big(\operatorname*{ess\,sup}_{0\leq s\leq t}|u(s)|\big).$$

Pick t small enough so that $1 - ML_2 t > 0$. Then

$$\sup_{0\leq s\leq t}|\phi(s,x,u) - \phi(s,x,0)| \quad \leq \quad \frac{Mt}{1-ML_2t}\sigma\big(\operatorname*{ess\,sup}_{0\leq s\leq t}|u(s)|\big).$$

Using this estimate in (3.3.3), taking the limit $t \to +0$ and recalling that $u(\cdot)$ is a right-continuous function, we obtain for all $x: |x| \leq r$ and all $u \in \mathcal{U}$: $\|u\|_\infty \leq r_2$ the LISS estimate

$$\dot{V}_u(x) \leq -\alpha(|x|) + LM\sigma\big(|u(0)|\big). \tag{3.3.4}$$

This shows that V is a LISS Lyapunov function for (3.0.1). □

An immediate consequence of Proposition 3.3.2 is the following characterization of LISS property:

Proposition 3.3.3. *Let Assumption 1.1.1 hold and let there exist $\sigma \in \mathcal{K}$ and $\rho > 0$ so that for all $v \in U$: $|v| \leq \rho$ and all $x \in \mathbb{R}^n$: $|x| \leq \rho$ we have*

$$|f(x,v) - f(x,0)| \leq \sigma(|v|).$$

Then for the system (3.0.1) the following properties are equivalent:

(i) 0-UAS

(ii) Existence of a Lipschitz continuous 0-UAS Lyapunov function

(iii) Existence of a Lipschitz continuous LISS Lyapunov function

(iv) LISS

Proof. The proposition is a consequence of Propositions 2.5.2, 3.3.2, 3.3.1 and of an obvious fact that LISS implies 0-UAS. □

Exercise 3.3.1. *Prove that for the linear systems*

$$\dot{x} = Ax + Bu$$

the notions of ISS and LISS are equivalent.

3.4 Linearization method

It is not hard to check whether a linear system (3.1.4) is ISS: it is enough to check whether matrix A is Hurwitz. In contrast, it is usually hard to prove that a nonlinear system is ISS, since the general methods for constructions of ISS Lyapunov functions do not exist, if the system is not linear.

However, if our goal is to prove merely local ISS of a system (3.0.1), then it is possible to apply a simple and a the same time powerful linearization method. The origins of this method go back to the father of stability theory - A. M. Lyapunov and the statement of a linearization method for nonlinear systems without controls can be found in any textbook on dynamical systems, see e.g. [85, p. 100]. Here we prove the ISS counterpart of this theorem for systems with external inputs.

Theorem 3.4.1. *Let in equation* (3.0.1)

$$f(x, u) = Bx + Cu + g(x, u),$$

for some $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times m}$ *and* $g(x, u) = o(|x| + |u|)$, *for* $|x| + |u| \to 0$. *If the linearized system*

$$\dot{x} = Bx + Cu \tag{3.4.1}$$

is ISS, then (3.0.1) *is LISS.*

Proof. System (3.4.1) is ISS, and consequently 0-GAS, therefore for any $Q > 0$ there exists according to Theorem 2.4.3 a matrix $P > 0$ such that $B^T P + PB = -Q$.

We prove, that $V : \mathbb{R}^n \to \mathbb{R}_+$, defined by $V(x) = x^T Px$ is a LISS-Lyapunov function for a system (3.0.1) for properly chosen gains. Let us compute the Lie derivative of V with respect to the system (3.0.1), using that $P = P^T$ and that $x^T PBx = x^T B^T P^T x = x^T B^T Px$.

$$\begin{aligned}\dot{V}(x) &= (\nabla V)^T f(x, u) = \left(Px + (x^T P)^T\right)^T (Bx + Cu + g(x, u))\\ &= x^T (P^T B + PB)x + 2x^T P(Cu + g(x, u))\\ &\leq -x^T Qx + k|x|\|P\|(\|C\||u| + |g(x, u)|).\end{aligned}$$

Here $k > 0$ is some constant, which depends on the chosen norm of the matrices $\|P\|$, $\|C\|$. Since $g(x, u) = o(|x| + |u|)$ for $|x| + |u| \to 0$, for each $w > 0$ we can find ρ, such that

$$|g(x, u)| \leq w \cdot (|x| + |u|), \quad \forall x : |x| \leq \rho,\ \forall u : |u| \leq \rho.$$

Using this inequality, we continue estimates

$$\dot{V}(x) \leq -x^T Qx + kw\|P\||x|^2 + k\|P\|(\|C\| + w)|x||u|.$$

Take $\chi(r) := \sqrt{r}$. Then for $|u| \leq |x|^2$ we have:

$$\dot{V}(x) \leq -\lambda_{min}(Q)|x|^2 + kw\|P\||x|^2 + k\|P\|(\|C\| + w)|x|^3,$$

where $\lambda_{min}(Q) > 0$ is the smallest eigenvalue of Q. Choosing w and ρ small enough, we will have in the right hand side some negative quadratic function of $|x|$ (remember that $PB + B^T P$ is a negative definite matrix). This proves that V is a LISS-Lyapunov function, and consequently, (3.0.1) is LISS. □

Remark 3.4.1. *It is possible to give another proof of the linearization theorem without using the Lyapunov function, see [18, Theorem 2, p. 19]. It exploits so-called 'fading-memory' estimates.*

3.5 Is it always possible to find an exponential Lyapunov function

Definition 3.5.1. *A Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an exponential ISS Lyapunov function, if there exist $\tilde{\psi}_1, \tilde{\psi}_2 \in \mathcal{K}_\infty$, $\mu > 0$ and $\chi \in \mathcal{K}$ such that*

$$\tilde{\psi}_1(|x|) \leq V(x) \leq \tilde{\psi}_2(|x|), \quad \forall x \in \mathbb{R}^n \tag{3.5.1}$$

and for any $\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m$ the following implication holds:

$$V(x) \geq \chi(|u|) \Rightarrow \nabla V \cdot f(x,u) \leq -\mu V(x). \tag{3.5.2}$$

Next theorem shows that existence of an ISS Lyapunov function is equivalent to existence of an exponential ISS Lyapunov function.

Theorem 3.5.1. *Let V be an ISS Lyapunov function. Then for any $\mu > 0$ there exists an exponential ISS Lyapunov function W with the same Lyapunov gain as V and with the decay coefficient μ in (3.5.2).*

Proof. Let V be an ISS Lyapunov function as in Definition 3.2.1 with the corresponding functions $\psi_1, \psi_2, \chi, \alpha$.

Pick any function $a \in \mathcal{K}$ satisfying $a'(0) = 0$ and

$$a(r) \leq \min\{r, \alpha(r)\}. \tag{3.5.3}$$

In particular, a possible choice is

$$a(r) = \frac{2}{\pi}\int_0^r \frac{\min\{s, \alpha(s)\}}{1+s^2} ds.$$

Next define $\rho \in \mathcal{K}$ so that

$$\rho(r) := \begin{cases} e^{\int_1^r \frac{\mu}{a(s)} ds} & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases}$$

Since $a(r) \leq r$, then $\frac{1}{a(r)} \geq \frac{1}{r}$ and thus $\int_0^1 \frac{1}{a(r)} dr \geq \int_0^1 \frac{1}{r} dr = +\infty$. This implies that

$$\lim_{r \to +0} \rho(r) = e^{\int_1^0 \frac{\mu}{a(s)} ds} = e^{-\int_0^1 \frac{\mu}{a(s)} ds} = 0$$

and thus the function ρ is continuous over $\mathbb{R}_+$.

Analogously,

$$\lim_{r \to +\infty} \rho(r) = e^{\int_1^\infty \frac{\mu}{a(s)} ds} = \infty.$$

From the construction it is clear that ρ is increasing and thus $\rho \in \mathcal{K}_\infty$. One can prove that ρ is a continuous function on $\mathbb{R}_+$ (for details see [66, Lemma 12]).

Now consider

$$W := \rho \circ V. \tag{3.5.4}$$

Clearly, W satisfies (3.5.1) with $\tilde{\psi}_1 = \rho \circ \psi_1$ and $\tilde{\psi}_2 = \rho \circ \psi_2$. Let us compute the Lie derivative of W:

$$\begin{aligned} \dot{W}(x) &= \frac{d\rho(r)}{dr}\Big|_{r=V(x)} \nabla V(x) f(x,u) \\ &= \frac{\mu}{a(V(x))} \rho(V(x)) \nabla V(x) f(x,u) \\ &\leq -\frac{\mu}{a(V(x))} W(x) \alpha(V(x)) \\ &\leq -\frac{\mu}{\alpha(V(x))} W(x) \alpha(V(x)) \\ &= -\mu W(x) \end{aligned}$$

This shows that W is an exponential ISS-Lyapunov function □

3.6 Attractivity notions for systems with inputs

The notions of 0-LS and 0-GS lead to the following notions for the systems with inputs.

Definition 3.6.1. *System (3.0.1) is*

- *locally stable (LS), if* $\exists \sigma, \gamma \in \mathcal{K}_\infty$ *and* $r > 0$ *such that* $\forall x \in B_r$, $\forall u \in \mathcal{U}$: $\|u\|_\infty \leq r$, $\forall t \geq 0$ *it holds*

$$|\phi(t,x,u)| \leq \sigma(|x|) + \gamma(\|u\|_\infty). \tag{3.6.1}$$

- *globally stable (GS), if it is locally stable with* $r = \infty$.
- *practically globally stable (pGS), if* $\exists \sigma, \gamma \in \mathcal{K}_\infty$ *and* $c > 0$ *such that* $\forall x \in \mathbb{R}^n$, $\forall u \in \mathcal{U}$ *and* $\forall t \geq 0$ *it holds*

$$|\phi(t,x,u)| \leq \sigma(|x|) + \gamma(\|u\|_\infty) + c. \tag{3.6.2}$$

Remark 3.6.1. *It is easy to see that the notion of pGS is equivalent to the boundedness property (BND), as defined in [71, p. 1285].*

Next we introduce new attractivity properties

Definition 3.6.2. (3.0.1) *has*

- *limit property (LIM), if* $\exists \gamma \in \mathcal{K}_\infty \cup \{0\}$, *such that*

$$\inf_{t \geq 0} |\phi(t,x,u)| \leq \gamma(\|u\|_\infty), \quad \forall x \in \mathbb{R}^n, \ \forall u \in \mathcal{U}.$$

- *asymptotic gain property (AG), if* $\exists \gamma \in \mathcal{K}_\infty \cup \{0\}$ *such that for all* $\varepsilon > 0$, *for all* $x \in \mathbb{R}^n$ *and for all* $u \in \mathcal{U}$ *there exist* $\tau_a = \tau_a(\varepsilon, x, u) < \infty$:

$$\forall t \geq \tau_a \quad \Rightarrow \quad |\phi(t,x,u)| \leq \varepsilon + \gamma(\|u\|_\infty). \tag{3.6.3}$$

- *strong asymptotic gain (sAG), if* $\exists \gamma \in \mathcal{K}_\infty \cup \{0\}$ *such that for all* $x \in \mathbb{R}^n$ *and for all* $\varepsilon > 0$ *there exist* $\tau_a = \tau_a(\varepsilon, x) < \infty$:

$$\forall t \geq \tau_a, \ \forall u \in \mathcal{U} \quad \Rightarrow \quad |\phi(t,x,u)| \leq \varepsilon + \gamma(\|u\|_\infty). \tag{3.6.4}$$

- *uniform asymptotic gain (UAG), if* $\exists \gamma \in \mathcal{K}_\infty \cup \{0\}$ *such that* $\forall \varepsilon, \delta > 0\ \exists \tau_a = \tau_a(\varepsilon, \delta) < \infty$:

$$\forall t \geq \tau_a,\ \forall u \in \mathcal{U},\ \forall x \in B_\delta \quad \Rightarrow \quad |\phi(t,x,u)| \leq \varepsilon + \gamma(\|u\|_\infty). \tag{3.6.5}$$

All three properties AG, sAG and UAG imply that all trajectories converge to the ball of radius $\gamma(\|u\|_\infty)$ around the origin as soon as $t \to \infty$. The difference between AG, sAG and UAG is in a kind of dependence of τ_a on states and inputs. In UAG systems this time depends (besides ε) only on the norm of the state, in sAG systems it depends on the state x (and may vary for the states with the same norm), but does not depend on u. In AG systems τ_a depends both on x and on u. For systems without inputs AG and sAG properties are reduced to 0-GATT, and UAG property is resolved to 0-UGATT.

The next example shows, that we should in general pay in order to obtain additional uniformity of AG. However, in this example this cost can be made arbitrarily small.

Example 3.6.1. *Consider a system*

$$\dot{x} = -\frac{1}{1+|u(t)|}x \tag{3.6.6}$$

The solution of (3.6.6) *is given by*

$$\begin{aligned} |\phi(t,x,u)| &= e^{-\int_0^t \frac{1}{1+|u(s)|}ds}|x| \\ &\leq e^{-\int_0^t \frac{1}{1+\|u\|_{\mathcal{U}}}ds}|x| \\ &= e^{-\frac{1}{1+\|u\|_{\mathcal{U}}}t}|x|. \end{aligned} \tag{3.6.7}$$

Taking the limits $t \to \infty$ *we see that the system has asymptotic gain* $\gamma \equiv 0$.

Next we show that (3.6.6) *is UAG. To this end we continue computations* (3.6.7) *to obtain for any* $r > 0$

$$\begin{aligned} |\phi(t,x,u)| &\leq e^{-\frac{1}{1+\|u\|_{\mathcal{U}}}t}|x| \\ &\leq e^{-\frac{1}{1+\max\{\|u\|_{\mathcal{U}},\frac{1}{r}|x|\}}t}\max\{r\|u\|_{\mathcal{U}},|x|\}. \end{aligned} \tag{3.6.8}$$

If $r\|u\|_{\mathcal{U}} \leq |x|$, *then* (3.6.8) *implies*

$$|\phi(t,x,u)| \leq e^{-\frac{1}{1+\frac{1}{r}|x|}t}|x|. \tag{3.6.9}$$

Otherwise, if $r\|u\|_{\mathcal{U}} \geq |x|$, *then* (3.6.8) *leads to*

$$|\phi(t,x,u)| \leq e^{-\frac{1}{1+\|u\|_{\mathcal{U}}}t}r\|u\|_{\mathcal{U}} \leq r\|u\|_{\mathcal{U}}. \tag{3.6.10}$$

Combining (3.6.9) *and* (3.6.10) *we obtain for all* $x \in \mathbb{R}^n$, $u \in \mathcal{U}$, $t \geq 0$ *and for any* $r > 0$ *that*

$$|\phi(t,x,u)| \leq e^{-\frac{1}{1+\frac{1}{r}|x|}t}|x| + r\|u\|_{\mathcal{U}}. \tag{3.6.11}$$

This shows that (3.6.6) *is ISS with arbitrarily small linear gain. It is easy to obtain from the last estimate that* (3.6.6) *is also UAG with arbitrarily small linear gain.*

On the other hand (3.6.6) *is not sAG with the gain* $\gamma \equiv 0$. *To see this pick* $\varepsilon := \frac{1}{2}$, $x = 1$ *and consider constant inputs* $u(\cdot) \equiv c$. *Then*

$$\phi(t,1,u) = e^{-\frac{1}{1+c}t}.$$

If (3.6.6) *would be sAG with the zero gain, then it would exist a time* τ_a, *which does not depend on* u, *so that for all* c *and all* $t \geq \tau_a$ *it holds that*

$$e^{-\frac{1}{1+c}t} \leq \frac{1}{2}.$$

But this is false since $e^{-\frac{1}{1+c}\tau_a}$ *monotonically increases to 1 as long as* $c \to \infty$. *Thus,* (3.6.6) *is not sAG with zero gain (and thus also not UAG with a zero gain).*

Exercise 3.6.1. *Prove that (3.6.6) is ISS by constructing a Lyapunov function. Hint: consider* $V(x) = x^4$.

Exercise 3.6.2. *Conduct an analysis as in Example 3.6.1 for the equation* $\dot{x} = -\zeta(|u(t)|)x$ *for any* $\zeta \in \mathcal{L}$.

3.7 ISS = UAG

In this section we prove that ISS is equivalent to UAG. We start this section with a simple criterion for LS

Lemma 3.7.1. *System (3.0.1) is LS if and only if the following property holds*

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall x \in \mathbb{R}^n : |x| \leq \delta, \ \forall u \in \mathcal{U} : \|u\|_\infty \leq \delta \ \Rightarrow \ \sup_{t \geq 0} |\phi(t, x, u)| \leq \varepsilon. \tag{3.7.1}$$

Proof. "⇐". Let (3.0.1) be LS. Then $\exists \sigma, \gamma \in \mathcal{K}_\infty$ and $r > 0$ such that $\forall x \in \mathbb{R}^n$, $\forall u \in \mathcal{U}$, $|x| \leq r$, $\|u\|_\infty \leq r$, $\forall t \geq 0$ an estimate (3.6.1) holds.

Take any $\varepsilon > 0$ and choose $\delta = \delta(\varepsilon) := \min\{\sigma^{-1}(\frac{\varepsilon}{2}), \gamma^{-1}(\frac{\varepsilon}{2})\}$. Clearly, with this δ property (3.7.1) follows.

"⇒". Let (3.7.1) hold. Define

$$\delta(\varepsilon) := \max\{s : \forall x \in \mathbb{R}^n : |x| \leq s, \quad \forall u \in \mathcal{U} : \|u\|_\infty \leq s \Rightarrow \sup_{t \geq 0} |\phi(t, x, u)| \leq \varepsilon\}.$$

Clearly (3.7.1) implies that $\delta(\cdot) \in \mathcal{K}$. Define $\gamma = \delta^{-1}$. Then

$$|\phi(t, x, u)| \leq \gamma(\max\{|x|, \|u\|_\infty\}) \leq \gamma(|x|) + \gamma(\|u\|_\infty),$$

which shows LS. □

Our next result is a simple restatement of global stability:

Proposition 3.7.1. *pGS + LS ⇔ GS*

Proof. The implication pGS + LS ⇐ GS is clear. Let us prove that pGS + LS ⇒ GS. Assume that (3.0.1) is LS and pGS. This means, that there exist $\sigma_1, \gamma_1, \sigma_2, \gamma_2 \in \mathcal{K}_\infty$ and $r, c > 0$ so that for all $x : |x| \leq r$ and all $u : \|u\|_\infty \leq r$ it holds that

$$|\phi(t, x, u)| \leq \sigma_1(|x|) + \gamma_1(\|u\|_\infty)$$

and for all $x \in \mathbb{R}^n$ and all $u \in \mathcal{U}$ the following estimate holds:

$$|\phi(t, x, u)| \leq \sigma_2(|x|) + \gamma_2(\|u\|_\infty) + c$$

Assume without restriction that $\sigma_2(s) \geq \sigma_1(s)$ and $\gamma_2(s) \geq \gamma_1(s)$ for all $s \geq 0$.

Pick $k_1, k_2 > 0$ so that $c = k_1\sigma_2(r)$ and $c = k_2\gamma_2(r)$. Then

$$c \leq c + c \leq k_1\sigma_2(|x|) + k_2\gamma_2(\|u\|_\infty)$$

for any $x \in \mathbb{R}^n$ and any $u \in \mathcal{U}$ so that either $|x| \geq r$ or $\|u\|_\infty \geq r$. Thus for all $x \in \mathbb{R}^n$ and all $u \in \mathcal{U}$ it holds that

$$|\phi(t, x, u)| \leq (1 + k_1)\sigma_2(|x|) + (1 + k_2)\gamma_2(\|u\|_\infty).$$

This shows global stability of (6.2.1). □

Lemma 3.7.2. *If (3.0.1) possesses UAG property, then (3.0.1) is GS.*

Proof. We are going to prove that UAG systems are LS and pGS. Then the claim will follow by Proposition 3.7.1.

In order to prove LS we will show the property (3.7.1).

Take any $\varepsilon > 0$. Let $\tau_a = \tau_a(\varepsilon/2, 1)$. Pick any $\delta_1 > 0$ so that $\gamma(\delta_1) < \varepsilon/2$. Then for all $x \in \mathbb{R}^n$: $|x| \leq 1$ and $\forall u \in \mathcal{U} : \|u\|_\infty \leq \delta_1$

$$\sup_{t \geq \tau_a} |\phi(t, x, u)| \leq \frac{\varepsilon}{2} + \gamma(\|u\|_\infty) < \varepsilon. \tag{3.7.2}$$

By continuity of ϕ with respect to controls and initial conditions, there is some $\delta_2 = \delta_2(\varepsilon, \tau_a) > 0$ so that

$$\forall \eta \in \mathbb{R}^n : |\eta| \leq \delta_2,\ \forall u \in \mathcal{U} : \|u\|_\infty \leq \delta_2 \Rightarrow \sup_{t \in [0, \tau_a]} |\phi(t, \eta, u)| \leq \varepsilon$$

Together with (3.7.2), this proves (3.7.1) with $\delta := \min\{1, \delta_1, \delta_2\}$.

Let (3.0.1) possess UAG property and let $\varepsilon := 1$ in the definition of UAG. Then for any $\delta > 0$ there exists $\tau_a = \tau_a(1, \delta)$: for all $x \in \mathbb{R}^n$: $|x| \leq \delta$ and for all $u \in \mathcal{U}$: $\|u\|_\infty \leq \delta$ it follows $|\phi(t, x, u)| \leq 1 + \gamma(\delta)$ for all $t \geq \tau_a$.

Define

$$K(\delta) := \sup\{|\phi(t, x, u)| : t \leq \tau_a,\ x \in \mathbb{R}^n :\ |x| \leq \delta,\ u \in \mathcal{U} :\ \|u\|_\infty \leq \delta\}.$$

From continuous dependence of solutions of (3.0.1) on initial data and on external inputs, it follows that K is well-defined and finite for any δ. Also $K(\cdot)$ is a nondecreasing function and $K(0) = 0$ since $x \equiv 0$ is an equilibrium of the undisturbed system. Define $\varphi(r) := r + \frac{1}{r}\int_r^{2r} K(s)ds$, which is continuous, strictly increasing to infinity and $\varphi(r) \geq K(r)$ for all $r \geq 0$.

Then for all $t \geq 0$, all $x \in \mathbb{R}^n$ and all $u \in \mathcal{U}$ we have

$$|\phi(t, x, u)| \leq 1 + \varphi(\max\{|x|, \|u\|_\infty\}) + \gamma(\max\{|x|, \|u\|_\infty\})$$

which can be rewritten by using $\sigma := \varphi + \gamma \in \mathcal{K}_\infty$ as

$$|\phi(t, x, u)| \leq 1 + \sigma(|x|) + \sigma(\|u\|_\infty)$$

which shows that (3.0.1) is pGS. Application of Proposition 3.7.1 finishes the proof. □

The final result of this section is:

Proposition 3.7.2. (3.0.1) *possesses UAG property* $\Leftrightarrow$ (3.0.1) *is ISS.*

Proof. Let us prove '$\Leftarrow$'. Let (3.0.1) be ISS. Take arbitrary $\varepsilon, \delta > 0$. Define $\tau_a = \tau_a(\varepsilon, \delta)$ as a solution of an equation $\beta(\delta, \tau_a) = \varepsilon$ (if it exists, then it is unique, because of monotonicity of β w.r.t. the second argument, if it does not exist, we put $\tau_a(\varepsilon, \delta) = 0$). Then for all $t \geq \tau_a$, all $x \in \mathbb{R}^n$: $|x| \leq \delta$ and all $u \in \mathcal{U}$

$$|\phi(t, x, u)| \leq \beta(|x|, t) + \gamma(\|u\|_\infty) \leq \beta(|x|, \tau_a) + \gamma(\|u\|_\infty) \leq \varepsilon + \gamma(\|u\|_\infty),$$

and the estimate (3.6.5) holds.

Let us prove '$\Rightarrow$'. Fix arbitrary $\delta \in \mathbb{R}_+$. We are going to construct function $\beta \in \mathcal{KL}$, so that (3.1.1) holds.

In Lemma 3.7.2 we have already proved that (3.0.1) is GS, that is $\forall t \geq 0, \forall x \in \mathbb{R}^n : |x| \leq \delta$ and $\forall u \in \mathcal{U}$ the following holds

$$|\phi(t, x, u)| \leq \sigma(\delta) + \gamma(\|u\|_\infty).$$

Define $\varepsilon_n := \frac{1}{2^n}\sigma(\delta)$, for $n \in \mathbb{N}$. UAG implies that there exists the sequence of times $\tau_n := T(\varepsilon_n, \delta)$ that we assume without loss of generality to be strictly increasing, such that

$$|\phi(t, x, u)| \leq \varepsilon_n + \gamma(\|u\|_\infty) \quad \forall x \in \mathbb{R}^n :\ |x| \leq \delta,\ \forall u \in \mathcal{U},\ \forall t \geq \tau_n.$$

Define $\omega(\delta, \tau_n) := \varepsilon_{n-1}$, for $n \in \mathbb{N}$, $n \neq 0$.

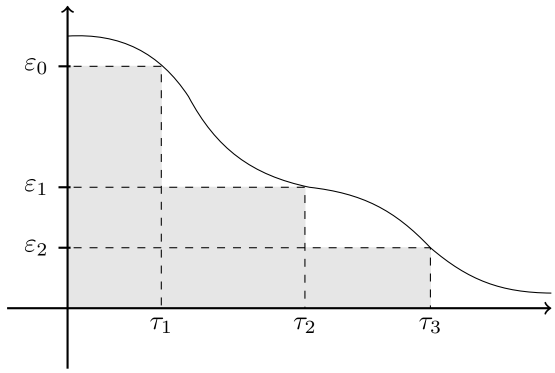

Figure 3.3: Construction of the function ω

Extend the function $\omega(\delta,\cdot)$ for $t \in \mathbb{R}_+\backslash\{\tau_n, n \in \mathbb{N}\}$ so that $\omega(\delta,\cdot) \in \mathcal{L}$ (see Figure 3.7). All such functions satisfy estimate (3.1.1), because for all $t \in (\tau_n, \tau_{n+1})$ it holds $|\phi(t,x,u)| \leq \varepsilon_n + \gamma(\|u\|_\infty) < \omega(\delta,t) + \gamma(\|u\|_\infty)$. Doing this for all $\delta \in \mathbb{R}_+$ we obtain function ω.

Now choose $\beta(r,t) = \sup_{0\leq s\leq r} \omega(s,t) \geq \omega(r,t)$. From this definition it follows that β is continuous and $\beta(\cdot,t) \in \mathcal{K}_\infty$. Also $\beta(r,\cdot) \in \mathcal{L}$, because $\omega(r,\cdot) \in \mathcal{L}$. Thus, $\beta \in \mathcal{KL}$ and estimate (3.1.1) is satisfied with such β. □

Exercise 3.7.1. *What is the relation between ISS with zero gain and UAG with zero gain?*

3.8 Characterizations of ISS

Having defined the needed notions we are able to give a precise formulation of our main result.

In their papers [70], [71] Sontag and Wang have shown the following fundamental result:

Theorem 3.8.1. *For the system* (3.0.1)*, satisfying Assumption 1.1.1 the combinations of properties depicted in Figure 3.4 are equivalent.*

In particular, ISS ⇔ UAG ⇔ AG+GS ⇔ AG+0-UGASs ⇔ existence of a Lyapunov function.

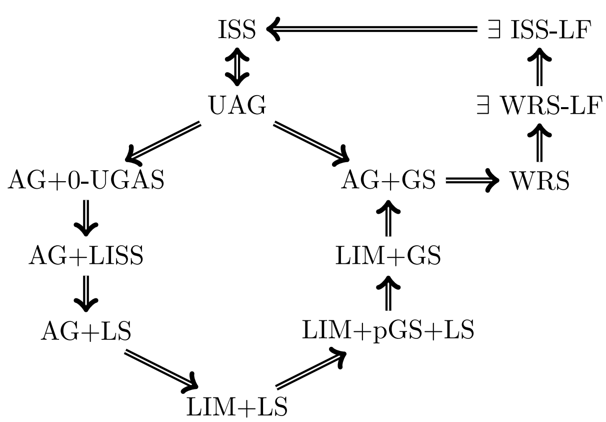

Figure 3.4: Characterizations of ISS in finite dimensions

3.9 Concluding remarks

The definition of input-to-state stability and of ISS Lyapunov function has been given in the original paper [68]. In the same paper it has been proved that existence of an ISS Lyapunov function implies ISS of a system.

Proposition 3.2.1 is [70, Remark 2.4]. This form of a Lyapunov function resembles the storage function in a theory of dissipative systems [82], [83].

The statement and proof of Theorem 3.5.1 is due to [66, Lemma 11]. This generalizes an earlier result [52, Theorem 3.6.10] for the GAS Lyapunov functions.

Chapter 4

Interconnections of ISS systems

Having studied properties of single control systems we proceed to investigation of interconnections of control systems. The main question in this respect is whether the system, consisting of stable (in some sense) components, is itself stable.

The most general type of interconnection of two systems is a feedback interconnection, depicted in Figure 4.1.

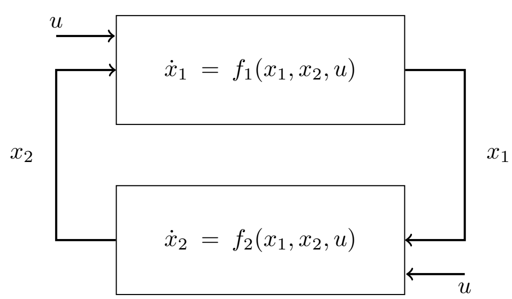

Figure 4.1: Feedback interconnection

In feedback interconnections the state of one system is transmitted as an input to another system and vice versa. Also the systems may have an additional external input u. The assumption that both subsystems have the same input does not restrict generality, since for any inputs to subsystems u_1 and u_2 one can introduce the composite input $u = (u_1, u_2)$, and assume that this input is transferred to both subsystems.

A feedback interconnection is called a cascade interconnection, if one subsystem evolves completely independently of another subsystem, but at the same time it influences the second subsystem, see Figure 4.2 (where x_2-subsystem is independent on x_1, but x_1 depends on x_2).

It is natural to ask, which properties should the subsystems possess so that the interconnection still has the same kind of stability. The next example shows that 0-GAS property is too weak to make it interesting for study of interconnections of control systems.

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -x_2 \end{aligned} \tag{4.0.1}$$

This system is a cascade interconnection of two 0-GAS systems, but for $x_1(0) \neq 0$ and $x_2(0)$ large enough the whole system (4.0.1) exhibits a finite escape time.

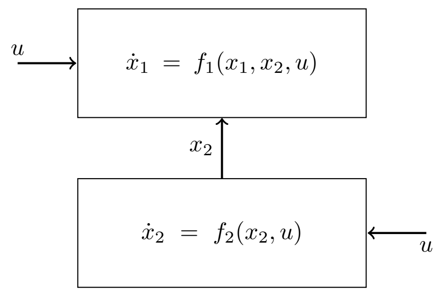

Figure 4.2: Cascade interconnection

In contrast, next result shows that cascade interconnections of ISS systems are always ISS.

Theorem 4.0.1. *Consider an interconnection*

$$\begin{aligned}\dot{x} &= f(x,z)\\ \dot{z} &= g(z,u)\end{aligned} \tag{4.0.2}$$

If both x-subsystem and z-subsystem are ISS, then the interconnected system (4.0.2) is also ISS.

This result can be obtained as a special case of the small-gain theorem, which we will prove later. But at this point we prefer a direct proof of this result.

Before we give a proof of this result, which is a bit technical, we show why the straightforward argument, which comes as the first to the brain, does not work.

Since x-subsystem is ISS, there exist $\beta_1 \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}_\infty$ so that for all $x \in \mathbb{R}^{n_1}$, all z and for all $t \geq 0$ it holds that

$$|\phi_x(t,x,z)| \leq \beta_1(|x|,t) + \gamma_1(\sup_{0\leq s\leq t} |z(s)|). \tag{4.0.3}$$

Analogously, ISS of the z-subsystem implies existence of $\beta_2 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}_\infty$ so that for all $z \in \mathbb{R}^{n_2}$, all u and for all $t \geq 0$ it holds that

$$|\phi_z(t,z,u)| \leq \beta_2(|z|,t) + \gamma_2(\sup_{0\leq s\leq t} |u(s)|). \tag{4.0.4}$$

However, we cannot obtain ISS of an interconnected system (4.0.2) just by plugging the estimate (4.0.4) into (4.0.3), since then we obtain

$$|\phi_x(t,x,z)| \leq \beta_1(|x|,t) + \gamma_1\big(\sup_{0\leq s\leq t} (\beta_2(|z|,s) + \gamma_2(\sup_{0\leq s\leq t} |u(s)|))\big).$$

Now for $u \equiv 0$ we obtain

$$|\phi_x(t,x,z)| \leq \beta_1(|x|,t) + \gamma_1(\beta_2(|z|,0)),$$

which is not a 0-GAS estimate. Thus, we should try another way.

Proof. Due to Proposition 1.1.3 we know that for the solution of the x-subsystem ϕ_x it holds that $\phi_x(t,x,z) = \phi_x\big(\frac{t}{2}, \phi_x(\frac{t}{2},x,z), z(\cdot+\frac{t}{2})\big)$.

Then ISS of x-subsystem implies that

$$\begin{aligned}|\phi_x(t,x,z)| &= \Big|\phi_x\Big(\tfrac{t}{2},\phi_x\big(\tfrac{t}{2},x,z(\cdot)\big),z\big(\,\cdot+\tfrac{t}{2}\big)\Big)\Big| \\ &\leq \beta_1\Big(|\phi_x\big(\tfrac{t}{2},x,z)|,\tfrac{t}{2}\Big)+\gamma_1\Big(\sup_{\frac{t}{2}\leq s\leq t}|z(s)|\Big) \\ &\leq \beta_1\Big(\beta_1\big(|x|,\tfrac{t}{2}\big)+\gamma_1\big(\sup_{0\leq s\leq \frac{t}{2}}|z(s)|\big),\tfrac{t}{2}\Big) \\ &\qquad +\gamma_1\Big(\sup_{\frac{t}{2}\leq s\leq t}|z\,(s)\,|\Big) \\ &\leq \beta_1\Big(2\beta_1\big(|x|,\tfrac{t}{2}\big),\tfrac{t}{2}\Big)+\beta_1\Big(2\gamma_1\big(\sup_{0\leq s\leq \frac{t}{2}}|z(s)|\big),\tfrac{t}{2}\Big) \\ &\qquad +\gamma_1\Big(\sup_{\frac{t}{2}\leq s\leq t}|z(s)|\Big).\end{aligned}$$

Now recall, that z-subsystem is ISS, and thus there exist $\beta_2\in\mathcal{KL}$ and $\gamma_2\in\mathcal{K}_\infty$ so that for all $z\in\mathbb{R}^{n_2}$, all u and for all $t\geq 0$ it holds that

$$|\phi_z(t,z,u)|\leq\beta_2(|z|,t)+\gamma_2(\sup_{0\leq s\leq t}|u(s)|).$$

This implies that

$$\sup_{0\leq s\leq t/2}|\phi_z(s,z,u)|\leq\beta_2(|z|,0)+\gamma_2(\sup_{0\leq s\leq t/2}|u(s)|). \tag{4.0.5}$$

Now we substitute $z(\cdot):=\phi_z(\cdot,x,u)$ into the previous computations. We obtain:

$$\begin{aligned}|\phi_x(t,x,z)| &\leq \beta_1\Big(2\beta_1\big(|x|,\tfrac{t}{2}\big),\tfrac{t}{2}\Big)+\beta_1\Big(2\gamma_1\big(\beta_2(|z|,0)+\gamma_2(\sup_{0\leq s\leq\frac{t}{2}}|u(s)|)\big),\tfrac{t}{2}\Big) \\ &\qquad +\gamma_1\Big(\sup_{\frac{t}{2}\leq s\leq t}\big(\beta_2(|z|,s)+\gamma_2(\sup_{0\leq r\leq s}|u(r)|)\big)\Big) \\ &\leq \beta_1(2\beta_1(|x|,\tfrac{t}{2}),\tfrac{t}{2})+\beta_1\Big(2\gamma_1\big(2\beta_2(|z|,0)\big)+2\gamma_1\big(2\gamma_2(\sup_{0\leq s\leq\frac{t}{2}}|u(s)|)\big),\tfrac{t}{2}\Big) \\ &\qquad +\gamma_1\Big(\sup_{\frac{t}{2}\leq s\leq t}\big(\beta_2(|z|,s)+\gamma_2(\sup_{0\leq r\leq s}|u(r)|)\big)\Big) \\ &\leq \beta_1(2\beta_1(|x|,\tfrac{t}{2}),\tfrac{t}{2})+\beta_1\Big(4\gamma_1\big(2\beta_2(|z|,0)\big),\tfrac{t}{2}\Big)+\beta_1\Big(4\gamma_1\big(2\gamma_2(\sup_{0\leq s\leq t}|u(s)|)\big),0\Big) \\ &\qquad +\gamma_1\Big(\beta_2(|z|,\tfrac{t}{2})+\gamma_2(\sup_{0\leq r\leq t}|u(r)|)\Big) \\ &\leq \beta_1(2\beta_1(|x|,\tfrac{t}{2}),\tfrac{t}{2})+\beta_1\Big(4\gamma_1\big(2\beta_2(|z|,0)\big),\tfrac{t}{2}\Big)+\beta_1\Big(4\gamma_1\big(2\gamma_2(\sup_{0\leq s\leq t}|u(s)|)\big),0\Big) \\ &\qquad +\gamma_1\Big(2\beta_2(|z|,\tfrac{t}{2})\Big)+\gamma_1\Big(2\gamma_2(\sup_{0\leq r\leq t}|u(r)|)\Big).\end{aligned}$$

Denote

$$\bar{\beta}_1(r,t):=\beta_1(2\beta_1(r,\tfrac{t}{2}),\tfrac{t}{2})+\beta_1\Big(4\gamma_1\big(2\beta_2(r,0)\big),\tfrac{t}{2}\Big)+\gamma_1\Big(2\beta_2(r,\tfrac{t}{2})\Big).$$

and

$$\bar{\gamma}_1(r):=\beta_1\Big(4\gamma_1\big(2\gamma_2(r)\big),0\Big)+\gamma_1\Big(2\gamma_2(r)\Big).$$

Then

$$|\phi_x(t,x,z)| \leq \bar{\beta}_1(|y|,t) + \bar{\gamma}_1(\sup_{0\leq s\leq t}|u(s)|).$$

Overall,

$$\begin{aligned}|\phi_y(t,y,u)| &\leq \bar{\beta}_1(|y|,t) + \bar{\gamma}_1(\sup_{0\leq s\leq t}|u(s)|)\\ &\qquad +\beta_2(|z|,t) + \gamma_2(\sup_{0\leq s\leq t}|u(s)|).\\ &\leq \bar{\beta}_1(|y|,t) + \beta_2(|y|,t)\\ &\qquad +\bar{\gamma}_1(\sup_{0\leq s\leq t}|u(s)|) + \gamma_2(\sup_{0\leq s\leq t}|u(s)|).\end{aligned}$$

This shows ISS of the overall system (4.0.2). □

4.1 Feedback interconnections

In this section we turn our attention to general feedback interconnections of n systems of the form

$$\begin{cases}\dot{x}_i = f_i(x_1,\ldots,x_n,u),\\ i=1,\ldots,n.\end{cases} \tag{4.1.1}$$

Here $u \in L_\infty(\mathbb{R}_+,\mathbb{R}^m)$ is an external input, $x_i(t) \in \mathbb{R}^{p_i}$, and f_i are Lipschitz continuous w.r.t. x_i uniform w.r.t. external inputs. We call x_i the state of the i-th subsystem, x_j, $j \neq i$ *the internal inputs*, i.e. the inputs from the other subsystems and u we call *the external input* to the system. The dimension of the state space of the whole system we denote as $N := p_1 + \ldots + p_n$.

The solution of the whole system (4.1.1) is an absolutely continuous function. Since globally bounded absolutely continuous functions belong to the space $L_\infty(\mathbb{R}_+,\mathbb{R}^N)$, we may consider that the whole input to the i-th subsystem $\tilde{x}_i := (x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n,u)$ belongs to the space $L_\infty(\mathbb{R}_+,\mathbb{R}^{N+m-p_i})$. Consequently, the i-th subsystem is a control system of the same kind as the whole system.

The system (4.1.1) can be viewed as a system

$$\begin{cases}\dot{x} = f(x,u),\ t>0\\ x(0)=x_0.\end{cases} \tag{4.1.2}$$

with the state $x := (x_1,\ldots,x_n)^T \in \mathbb{R}^N$, input $u \in L_\infty(\mathbb{R}_+,\mathbb{R}^m)$ and with the nonlinear dynamics $f(x,u) := (f_1^T(x,u),\ldots,f_n^T(x,u))^T$.

The state of the i-th subsystem at time $t \geq 0$ corresponding to the initial condition $x_i \in \mathbb{R}^{p_i}$ and total input $\tilde{x}_i(\cdot)$ we denote by $\phi_i(t,x_i,\tilde{x}_i)$.

We are going to derive conditions guaranteeing that a system, consisting of ISS components is itself ISS.

According to Definition 3.1.1, the i-th subsystem of the system (4.1.1) is ISS if there exist $\beta_i \in \mathcal{KL}$ and $\gamma_i \in \mathcal{K}$ such that $\forall x_i \in \mathbb{R}^{p_i}$, $\forall \tilde{x}_i \in L_\infty(\mathbb{R}_+,\mathbb{R}^{N+m-p_i})$ and $\forall t \geq 0$ the following holds

$$|\phi_i(t,x_i,\tilde{x}_i)| \leq \beta_i(|x_i|,t) + \gamma_i(\|\tilde{x}_i(\cdot)\|_\infty). \tag{4.1.3}$$

This definition makes possible to measure how the state of the i-th subsystem depends on the magnitude of the total input to the i-th subsystem, but this form is not suitable to measure how the state of the i-th subsystem depends on the magnitude of particular internal inputs x_j.

The following restatement of the ISS property for the i-th subsystem will be exploited a lot on the following pages:

Proposition 4.1.1. *The i-th subsystem of (4.1.1) is ISS (in maximum formulation) if and only if there exist $\gamma_{ij},\ \gamma_i \in \mathcal{K}$, $j = 1, \dots, n$, $j \neq i$ and $\beta_i \in \mathcal{KL}$, such that for all initial values x^0 and all inputs u the inequality*

$$|\phi_i(t, x_i, \tilde{x}_i)| \leq \max \left\{ \beta_i \left(\left| x_i^0 \right|, t \right), \max_{j \neq i} \gamma_{ij} \left(\|x_j\|_\infty \right), \gamma_i \left(\|u_i\|_\infty \right) \right\} \tag{4.1.4}$$

is satisfied $\forall t \in \mathbb{R}_+$. The functions γ_{ij} and γ_i are called (nonlinear) gains.

If instead of inequality (4.1.4) the inequality

$$|\phi_i(t, x_i, \tilde{x}_i)| \leq \beta_i \left(\left| x_i^0 \right|, t \right) + \sum_{j \neq i} \gamma_{ij} \left(\|x_j\|_\infty \right) + \gamma_i \left(\|u_i\|_\infty \right) \tag{4.1.5}$$

holds, then the i-th subsystem of (4.1.1) is ISS in summation formulation.

Proof. We prove the statement concerning sum-formulation. The max-formulation can be proved analogously.

Let there exist $\beta_i \in \mathcal{KL}$ and $\tilde{\gamma}_i \in \mathcal{K}$ so that $\forall x_i \in \mathbb{R}^{p_i}$, $\forall \tilde{x}_i \in L_\infty(\mathbb{R}_+, \mathbb{R}^{N+m-p_i})$ and $\forall t \geq 0$ the inequality (4.1.3) holds.

According to the definition, $\|\tilde{x}_i(\cdot)\|_\infty = \sup_{t \geq 0} |\tilde{x}_i(t)|$. Due to equivalence of norms in finite-dimensional spaces, there exists $m, M > 0$: for all $t \geq 0$ it holds that

$$m \max\{|u(t)|, \max_{j \neq i} |x_j(t)|\} \leq |\tilde{x}_i(t)| \leq M \max\{|u(t)|, \max_{j \neq i} |x_j(t)|\}. \tag{4.1.6}$$

Thus we can proceed to

$$\begin{aligned}
\tilde{\gamma}_i(\|\tilde{x}_i(\cdot)\|_\infty) &= \sup_{t \geq 0} \tilde{\gamma}_i(|\tilde{x}_i(t)|) \\
&\leq \sup_{t \geq 0} \tilde{\gamma}_i\big(M \max\left\{|u(t)|, \max_{j \neq i} |x_j(t)|\right\}\big) \\
&= \tilde{\gamma}_i\big(M \max\left\{\|u\|_\infty, \max_{j \neq i} \|x_j\|_\infty\right\}\big) \\
&= \max\left\{\tilde{\gamma}_i(M\|u\|_\infty), \max_{j \neq i} \tilde{\gamma}_i(M\|x_j\|_\infty)\right\} \\
&= \max\left\{\tilde{\gamma}_i(M\|u\|_\infty), \max_{j \neq i} \tilde{\gamma}_i(M\|x_j\|_\infty)\right\} \\
&\leq \max\left\{\tilde{\gamma}_i(M\|u\|_\infty), \max_{j \neq i} \tilde{\gamma}_i(M\|x_j\|_\infty)\right\} \\
&= \sum_{j \neq i} \gamma_{ij}(\|x_j\|_\infty) + \gamma_i(\|u\|_\infty),
\end{aligned}$$

where $\gamma_{ij}(r) = \gamma_i(r) = \tilde{\gamma}_i(Mr)$, for all $r \geq 0$ and all $j \neq i$.

All other claims of the theorem are shown analogously. □

If the system is ISS in summation formulation, then it is ISS also in maximum formulation and vice versa, however, the gains can be different.

4.2 Small-gain theorem in terms of Lyapunov functions

Let us assume that all subsystems x_i for $i = 1, \dots, n$ are ISS and thus, according to converse ISS Lyapunov theorem (Theorem 3.2.3) every subsystem possesses a smooth ISS-Lyapunov function w.r.t. the full input to the i-th subsystem $\tilde{x}_i := (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n, u)$.

According to Definition 3.2.1, a smooth function $V_i : \mathbb{R}^{p_i} \to \mathbb{R}_+$ is an ISS-Lyapunov function (ISS-LF) for the i-th subsystem of (4.1.1), if and only if there exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, $h_i \in \mathcal{K}$ and a positive definite function α_i, such that:

$$\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|), \quad \forall x_i \in \mathbb{R}^{p_i} \tag{4.2.1}$$

and $\forall x_i \in \mathbb{R}^{p_i}$, $\forall \tilde{x}_i \in L_\infty(\mathbb{R}_+, \mathbb{R}^{N+m-p_i})$ it holds for

$$V_i(x_i) \geq h_i(|\tilde{x}_i|) \;\Rightarrow\; \nabla V_i(x_i) \cdot f_i(x_1, \dots, x_n, u) \leq -\alpha_i(V_i(x_i)). \tag{4.2.2}$$

In this form the ISS Lyapunov functions will not be very helpful for the analysis of interconnections. Next we restate a notion of an ISS-Lyapunov function of the i-th subsystem of (4.1.1) in a more useful way.

Proposition 4.2.1. *A smooth function $V_i : \mathbb{R}^{p_i} \to \mathbb{R}_+$ is an ISS-Lyapunov function (ISS-LF) for the i-th subsystem of (4.1.1), if and only if there exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, $\chi_{ij}, \chi_i \in \mathcal{K}$, $j = 1, \dots, n$, $j \neq i$, $\chi_{ii} := 0$ and a positive definite function α_i, such that (4.2.1) holds and $\forall x_j \in \mathbb{R}^{p_j}$, $j = 1, \dots, n$ and $\forall u \in \mathbb{R}^m$ it holds the implication*

$$V_i(x_i) \geq \max\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(|u|)\} \;\Rightarrow\; \nabla V_i(x_i) \cdot f_i(x_1, \dots, x_n, u) \leq -\alpha_i(V_i(x_i)). \tag{4.2.3}$$

Proof. Let V_i be an ISS Lyapunov function for the i-th subsystem so that (4.2.1) and (4.2.2) hold.

Due to (4.1.6) we have that

$$\begin{aligned} h_i(|\tilde{x}_i|) &\leq \max\{\max_{j=1, j\neq i}^{n}\{h_i(M|x_j|)\}, h_i(M|\xi|)\} \\ &\leq \max\{\max_{j=1, j\neq i}^{n}\{h_i(M\psi_{j1}^{-1}(V_j(x_j))\}, h_i(M|\xi|)\} \end{aligned}$$

Define the gains as follows:

$$\chi_{ij}(r) := h_i(M\psi_{i1}^{-1}(r)), \quad \chi_i(r) := h_i(Mr),\; i \neq j,\; r \geq 0.$$

Then

$$V_i(x_i) \geq \max\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(|u|)\}$$

implies

$$V_i(x_i) \geq h_i(|\tilde{x}_i|)$$

which due to (4.2.2) results in

$$\nabla V_i(x_i) \cdot f_i(x_1, \dots, x_n, u) \leq -\alpha_i(V_i(x_i)).$$

Thus, (4.2.3) holds.

Now let (4.2.3) holds.

Define

$$h_i(r) := \max\{\max_{j=1}^{n} \chi_{ij}(\psi_{j2}(\tfrac{r}{m})), \chi_i(\tfrac{r}{m})\},$$

where m comes from (4.1.6). Then

$$V_i(x_i) \geq h_i(|\tilde{x}_i|)$$

implies due to (4.1.6)

$$\begin{aligned} V_i(x_i) &\geq \max\{\max_{j=1}^{n} \chi_{ij}(\psi_{j2}(\tfrac{|\tilde{x}_i|}{m})), \chi_i(\tfrac{|\tilde{x}_i|}{m})\} \\ &\geq \max\{\max_{j=1}^{n} \chi_{ij}(\psi_{j2}(|x_j|)), \chi_i(|\xi|)\} \\ &\geq \max\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(|\xi|)\}, \end{aligned}$$

which implies due to (4.2.3) the inequality

$$\nabla V_i(x_i) \cdot f_i(x_1, \dots, x_n, u) \leq -\alpha_i(V_i(x_i)).$$

This shows the claim of the proposition. □

The form of a definition of an ISS Lyapunov function for the subsystems, introduced in Proposition 4.2.1 helps to distinguish between the influence of different subsystems on other ones. Functions χ_{ij}, $i, j = 1, \dots, n$ are called *internal Lyapunov gains* and χ_i, $i = 1, \dots, n$ are called *external Lyapunov gains*. The internal Lyapunov gains χ_{ij} characterize the interconnection structure of subsystems. As we will see, the question, whether the interconnection (4.1.1) is ISS, depends on the properties of the gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(s) := \left(\max_{j=1}^{n} \chi_{1j}(s_j), \dots, \max_{j=1}^{n} \chi_{nj}(s_j) \right), \; s \in \mathbb{R}^n_+, \tag{4.2.4}$$

where χ_{ij} come from the formulation of Proposition 4.2.1.

To construct an ISS-Lyapunov function for the whole interconnection we will use the notion of Ω-path (see [20]).

Definition 4.2.1. *A function $\sigma = (\sigma_1, \dots, \sigma_n)^T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, where $\sigma_i \in \mathcal{K}_\infty$, $i = 1, \dots, n$ is called an Ω-path (with respect to operator Γ), if it possesses the following properties:*

1. *σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;*
2. *for every compact set $P \subset (0, \infty)$ there are finite constants $0 < K_1 < K_2$ such that for all points of differentiability of σ_i^{-1} we have*
$$0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2, \quad \forall r \in P;$$
3. $$\Gamma(\sigma(r)) < \sigma(r), \; \forall r > 0. \tag{4.2.5}$$

The next theorem provides a construction of an ISS-Lyapunov function for an interconnection of ISS subsystems, provided an Ω-path for the operator Γ is available, [21], [20].

Theorem 4.2.2. *Let for the i-th subsystem of (4.1.1) V_i be the ISS-Lyapunov function with corresponding gains χ_{ij}, $i = 1, \dots, n$. If there exists an Ω-path $\sigma = (\sigma_1, \dots, \sigma_n)^T$ corresponding to the operator Γ defined by (4.2.4), then the ISS-Lyapunov Lyapunov function for the system (4.1.1) can be constructed as*

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}, \tag{4.2.6}$$

The Lyapunov gain of the whole system is given by

$$\chi(r) := \max_i \sigma_i^{-1}(\chi_i(r)). \tag{4.2.7}$$

Proof. In order to prove that V is a Lyapunov function it is suitable to divide its domain of definition into subsets on which V takes a simpler form. Thus, for all $i \in \{1, \dots, n\}$ define a set

$$M_i = \left\{ x \in \mathbb{R}^n : \sigma_i^{-1}(V_i(x_i)) > \sigma_j^{-1}(V_j(x_j)), \; \forall j = 1, \dots, n, \; j \neq i \right\}.$$

From the continuity of V_i and σ_i^{-1}, $i = 1, \dots, n$ it follows that all M_i are open. Also note that $\mathbb{R}^n = \cup_{i=1}^n \overline{M}_i$ and for all $i \neq j$ holds $M_i \cap M_j = \emptyset$. Define

$$\gamma(r) := \max_{j=1}^{n} \sigma_j^{-1} \circ \gamma_j(r).$$

Take some $i \in \{1,\dots,n\}$ and pick any $x \in M_i$. Assume that $V(x) \geq \gamma(\|\xi\|_U)$ holds. Then we obtain

$$\sigma_i^{-1}(V_i(x_i)) = V(x) \geq \gamma(\|\xi\|_U) = \max_{j=1}^{n} \sigma_j^{-1} \circ \gamma_j(\|\xi\|_U) \geq \sigma_i^{-1}(\gamma_i(\|\xi\|_U)).$$

Since $\sigma_i^{-1} \in \mathcal{K}_\infty$ it holds

$$V_i(x_i) \quad \geq \quad \gamma_i(\|\xi\|_U). \tag{4.2.8}$$

On the other hand, from the condition (4.2.5) we obtain that

$$\begin{aligned} V_i(x_i) = \sigma_i(V(x)) \quad &\geq \quad \max_{j=1}^{n} \chi_{ij}\left(\sigma_j\left(V\left(x\right)\right)\right) = \max_{j=1}^{n} \chi_{ij}\left(\sigma_j\left(\sigma_i^{-1}\left(V_i\left(x_i\right)\right)\right)\right) \\ &> \quad \max_{j=1}^{n} \chi_{ij}\left(\sigma_j\left(\sigma_j^{-1}\left(V_j\left(x_j\right)\right)\right)\right) = \max_{j=1}^{n} \chi_{ij}\left(V_j\left(x_j\right)\right). \end{aligned}$$

Combining with (4.2.8) we obtain

$$V_i(x_i) \geq \max\left\{\max_{j=1}^{n} \chi_{ij}\left(V_j\left(x_j\right)\right), \gamma_i(\|\xi\|_U)\right\} \tag{4.2.9}$$

Hence condition (4.2.3) implies that for all x the following estimate holds

$$\begin{aligned} \frac{d}{dt}V(x) \quad &= \quad \frac{d}{dt}(\sigma_i^{-1}(V_i(x_i))) = \left(\sigma_i^{-1}\right)'(V_i(x_i))\frac{d}{dt}V_i(x_i(t)) \\ &\leq \quad -\left(\sigma_i^{-1}\right)'(V_i(x_i))\alpha_i(V_i(x_i)) = -\left(\sigma_i^{-1}\right)'(\sigma_i(V(x)))\alpha_i(\sigma_i(V(x))). \end{aligned}$$

We set

$$\alpha(r) := \min_{i=1}^{n}\left\{\left(\sigma_i^{-1}\right)'(\sigma_i(r))\alpha_i(\sigma_i(r))\right\}.$$

Function α is positive definite, because $\sigma_i^{-1} \in \mathcal{K}_\infty$ and all α_i are positive definite functions. Overall, for all $x \in \cup_{i=1}^{n} M_i$ holds

$$\frac{d}{dt}V(x) \leq -\min_{i=1}^{n}\left(\sigma_i^{-1}\right)'(\sigma_i(V(x)))\alpha_i(\sigma_i(V(x))) = -\alpha(V(x)).$$

Now let $x \notin \cup_{i=1}^{n} M_i$. From $X = \cup_{i=1}^{n} \overline{M}_i$ it follows that $x \in \cap_{i\in I(x)}\partial M_i$ for some index set $I(x) \subset \{1,\dots,n\}$, $|I(x)| \geq 2$.

$$\cap_{i\in I(x)}\partial M_i = \{x \in X : \forall i \in I(x),\ \forall j \notin I(x)\ \sigma_i^{-1}(V_i(x_i)) > \sigma_j^{-1}(V_j(x_j)),$$

$$\forall i,j \in I(x)\ \sigma_i^{-1}(V_i(x_i)) = \sigma_j^{-1}(V_j(x_j))\}.$$

Due to continuity of ϕ we have, that for all $u \in PC(\mathbb{R}_+,U)$, $u(0) = \xi$ there exists $t^* > 0$, such that for all $t \in [0,t^*)$ it follows $\phi(t,x,u) \in \left(\cap_{i\in I(x)}\partial M_i\right) \cup \left(\cup_{i\in I(x)} M_i\right)$.

Then, by definition of the Lie derivative we obtain

$$\begin{aligned} \dot{V}(x) \quad &= \quad \overline{\lim_{t\to+0}}\ \frac{1}{t}(V(\phi(t,x,u))) - V(x)) \\ &= \quad \overline{\lim_{t\to+0}}\ \frac{1}{t}\left(\max_{i\in I(x)}\{\sigma_i^{-1}(V_i(\phi_i(t,x,u)))\} - \max_{i\in I(x)}\{\sigma_i^{-1}(V_i(x_i))\}\right) \end{aligned} \tag{4.2.10}$$

From the definition of $I(x)$ it follows that

$$\sigma_i^{-1}(V_i(x_i)) = \sigma_j^{-1}(V_j(x_j)) \quad \forall i,j \in I(x),$$

and therefore the index i, on which we maximum of $\max_{i\in I(x)}\{\sigma_i^{-1}(V_i(x_i))\}$ is reached, may be always set equal to the index on which the maximum $\max_{i\in I(x)}\{\sigma_i^{-1}(V_i(\phi_i(t,x,u)))\}$ is reached.
We continue the estimates (4.2.10)

$$\dot{V}(x) = \overline{\lim_{t\to+0}} \max_{i\in I(x)} \{\frac{1}{t}\left(\sigma_i^{-1}(V_i(\phi_i(t,x,u))) - \sigma_i^{-1}(V_i(x_i))\right)\}$$

Using Lemma 1.4.3 we obtain

$$\begin{aligned}\dot{V}(x) &= \max_{i\in I(x)} \{\overline{\lim_{t\to+0}} \frac{1}{t}\left(\sigma_i^{-1}(V_i(\phi_i(t,x,u))) - \sigma_i^{-1}(V_i(x_i))\right)\} \\ &= \max_{i\in I(x)} \frac{d}{dt}(\sigma_i^{-1}(V_i(x_i))) \leq -\alpha(V(x)).\end{aligned}$$

Overall, we have that for all $x \in X$ holds

$$\frac{d}{dt}V(x) = \max_i\{\left(\sigma_i^{-1}\right)'(V_i(x_i))\frac{d}{dt}V_i(x_i(t))\} \leq -\alpha(V(x)),$$

and the ISS-Lyapunov function for the whole interconnection is constructed. ISS of the whole system follows by Theorem 3.2.3. □

In order to apply Theorem 4.2.2 one has to construct the Ω-path or at least prove its existence. To this end we introduce another notion: we say that Γ satisfies *the small-gain condition* if the following inequality holds

$$\Gamma(s) \not\geq s, \ \forall \ s \in \mathbb{R}^n_+ \setminus \{0\}\,. \tag{4.2.11}$$

Here for two vectors $v, w \in \mathbb{R}^n_+$ the expression $v \geq w$ means that $v_i \geq w_i$ for all $i = 1, \ldots, n$. The relation $\leq$ on $\mathbb{R}^n_+$ is defined analogously. Under '$\not\geq$' we understand a logical negation of '$\geq$'.

The next proposition shows that small-gain condition (4.2.11) holds if and only if all the cycles of gains are contractions.

Proposition 4.2.3. *Small-gain condition* (4.2.11) *holds if and only if for each cycle in* Γ *(that is for all* $(k_1, ..., k_p) \in \{1, ..., n\}^p$*, where* $k_1 = k_p$*) and for all* $s > 0$ *it holds that*

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ \ldots \circ \gamma_{k_{p-1}k_p}(s) < s. \tag{4.2.12}$$

Proof. Let all the cycles of gains be contractions, i.e. the condition (4.2.12) holds. Assume that the small-gain condition (4.2.11) does not hold, i.e. there exists certain $s \in \mathbb{R}^n_+$ so that $s \neq 0$ and

$$\Gamma(s) = \left(\max_{j=1}^{n} \chi_{1j}(s_j), \ldots, \max_{j=1}^{n} \chi_{nj}(s_j)\right) \geq s. \tag{4.2.13}$$

Then there exists a sequence $\{i_k\}$, $k = 1, \ldots, n$:

$$\chi_{ki_k}(s_{i_k}) \geq s_k.$$

Since $i_k \neq k$ for all k, there exist $p \in \{2, \ldots, n\}$ and a sequence $\{j_k\}$, $k = 1, \ldots, p$, so that $j_p = j_1$ and for all $k = 1, \ldots, p-1$ it holds that

$$\chi_{j_kj_{k+1}}(s_{j_{k+1}}) \geq s_{j_k}.$$

Due to these inequalities and since $\chi_{ij} \in \mathcal{K}$ for all $i, j : i \neq j$ the following derivations hold

$$\begin{aligned}&\chi_{j_1j_2} \circ \chi_{j_2j_3} \circ \ldots \circ \chi_{j_{p-2}j_{p-1}} \circ \chi_{j_{p-1}j_p}(s_{j_p}) \\ &\quad \geq \chi_{j_1j_2} \circ \chi_{j_2j_3} \circ \ldots \circ \chi_{j_{p-2}j_{p-1}}(s_{p-1}) \\ &\quad \vdots \\ &\quad \geq \chi_{j_1j_2}(s_{j_2}) \\ &\quad \geq s_{j_1}.\end{aligned}$$

Since $j_p = j_1$, we have that

$$\chi_{j_1 j_2} \circ \chi_{j_2 j_3} \circ \ldots \circ \chi_{j_{p-2} j_{p-1}} \circ \chi_{j_{p-1} j_1}(s_{j_1}) \geq s_{j_1}$$

This inequality contradicts to the assumption that all cycles of gains are contractions. Thus, small-gain condition (4.2.11) must hold.

Let us prove the converse statement. Assume that small-gain condition (4.2.11) holds. Then for any $\tilde{\Gamma} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, satisfying $\tilde{\Gamma} \leq \Gamma$ it also holds that

$$\tilde{\Gamma}(s) \not\geq s, \quad s \in \mathbb{R}^n_+ \backslash \{0\}. \tag{4.2.14}$$

Indeed, if there exist $s \in \mathbb{R}^n_+$: $\tilde{\Gamma}(s) \geq s$, then also $\Gamma(s) \geq s$, which contradicts to (4.2.11).

Pick $\tilde{\Gamma}$ of the special form:

$$(\tilde{\Gamma}(s))^T := (\gamma_{13}(s_3), \gamma_{21}(s_1), \gamma_{32}(s_2), \max_{j=1}^{n} \chi_{4j}(s_j), \ldots, \max_{j=1}^{n} \chi_{nj}(s_j)).$$

Clearly, $\tilde{\Gamma} \leq \Gamma$. Pick now a vector s of the special form

$$s := (r, \gamma_{21}(r), \gamma_{32} \circ \gamma_{21}(r), 0, \ldots, 0)^T,$$

where $r > 0$.

For this s the vector $\tilde{\Gamma}(s)$ looks like

$$\tilde{\Gamma}(s) := (\gamma_{32} \circ \gamma_{21} \circ \gamma_{13}(r), \gamma_{21}(r), \gamma_{32} \circ \gamma_{21}(r), (\tilde{\Gamma}(s))_4, \ldots, (\tilde{\Gamma}(s))_n)^T,$$

By construction $(\tilde{\Gamma}(s))_i \geq s_i$ for $i \geq 2$. Thus, due to (4.2.14) the condition

$$\gamma_{32} \circ \gamma_{21} \circ \gamma_{13}(r) < r$$

for all $r > 0$ must hold.

Analogously one can show that other cycles must be less than identity. □

Both formulations of the small-gain condition are frequently used in theoretical works. For applications the cyclic formulation seems to be more convenient.

Exercise 4.2.1. ***Restatements of small-gain conditions.***

1. *Let $\gamma_1, \gamma_2 \in \mathcal{K}$. Then $\gamma_1 \circ \gamma_2 < id \;\Leftrightarrow\; \gamma_2 \circ \gamma_1 < id$.*
2. *What equivalent representations exist in the case, when $\gamma_1 \circ \gamma_2 \circ \ldots \circ \gamma_n < id$?*

Exercise 4.2.2. *Prove, that in the linear 2-dimensional case ($\Gamma \in \mathbb{R}^{2\times 2}$) it holds $\gamma_{12} \cdot \gamma_{21} < 1 \;\Leftrightarrow\; \rho(\Gamma) < 1$, where*

$$\Gamma = \begin{pmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{pmatrix}$$

4.2.1 Examples

We conclude this section with two examples.

Example 4.2.4. *Let us consider the following system*

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + a\sqrt{|x_1|} \end{aligned} \tag{4.2.15}$$

We analyze GAS of this system with the help of small-gain theorems. Pick

$$V_1(x_1) := x_1^2. \tag{4.2.16}$$

as an ISS Lyapunov function candidate for the first subsystem. We obtain:

$$\dot{V}_1(x_1) = 2x_1\dot{x}_1 = 2x_1(-x_1 + x_2^2). \tag{4.2.17}$$

The Lyapunov gain χ_{12} *we choose as* $\chi_{12}(r) := \frac{1}{1-\varepsilon}r^2$ *for some* $\varepsilon \in (0,1)$*. Then* $|x_1| \geq \chi_{12}(|x_2|)$ *implies that* $x_2^2 \leq (1-\varepsilon)|x_1|$ *which immediately leads to*

$$\dot{V}_1(x_1) = -2\varepsilon x_2^2 = -2\varepsilon V_1(x_1). \tag{4.2.18}$$

This shows that the first subsystem is ISS with V_1 *as an ISS Lyapunov function.*

Now pick the following ISS Lyapunov function candidate for the second subsystem

$$V_2(x_2) := x_2^2. \tag{4.2.19}$$

We have:

$$\dot{V}_2(x_2) = 2x_2\dot{x}_2 = 2x_2(-x_2 + a\sqrt{|x_1|}). \tag{4.2.20}$$

Pick $\chi_{21}(r) := \frac{a}{1-\varepsilon_2}\sqrt{r}$*, for some* $\varepsilon_2 \in (0,1)$*. Then* $|x_2| \geq \chi_{21}(|x_1|)$ *implies that*

$$\dot{V}_2(x_2) = -2\varepsilon_2 x_2^2 = -2\varepsilon_2 V_2(x_2). \tag{4.2.21}$$

Thus, the second subsystem is also ISS.

To prove ISS of the interconnection we exploit the small-gain condition $\chi_{12} \circ \chi_{21}(r) < r$ *for all* $r > 0$*. In our case this reduces to*

$$\chi_{12} \circ \chi_{21}(r) = \frac{1}{1-\varepsilon}\frac{a^2}{(1-\varepsilon_2)^2}r < r,$$

for some $\varepsilon, \varepsilon_2 > 0$ *and all* $r > 0$*. Clearly, this is equivalent to* $|a| < 1$*.*

If $a \in \{1, -1\}$*, then the points satisfying algebraic equations*

$$\begin{aligned} x_1 &= x_2^2 \\ x_2 &= a\sqrt{|x_1|} \end{aligned}$$

are the stationary points of (4.2.15)*, which tells us that for* $a = 1, -1$ (4.2.15) *is not ISS. Using monotonicity arguments one can show that for* $a : |a| > 1$ *the system* (4.2.15) *is again not ISS.*

To conclude our investigations, (4.2.15) *is ISS iff* $|a| < 1$*.* □

Example 4.2.5. *Next we consider a dynamical system which can arise e.g. in modeling of chemical reactions or production networks [15]*

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{n} c_{ij}(x(t))\tilde{f}_j(x_j(t)) + u_i(t) - \tilde{c}_{ii}(x(t))\tilde{f}_i(x_i(t)), \quad i = 1, \ldots, n. \tag{4.2.22}$$

Denoting $c_{ii} := -\tilde{c}_{ii}$ *we can rewrite the above equations in a vector form*

$$\dot{x}(t) = C(x(t))\tilde{f}(x(t)) + u(t), \tag{4.2.23}$$

where $\tilde{f}(x(t)) = (\tilde{f}_1(x_1(t)), ..., \tilde{f}_n(x_n(t)))^T$*,* $u(t) = (u_1(t), \ldots, u_n(t))^T$ *and* $C(x) \in \mathbb{R}^{n\times n}$*.*

In order to analyze stability of the system (4.2.23) we are going to exploit Theorem 4.2.2. We construct ISS-Lyapunov functions $V_i(x_i)$ and corresponding gains χ_{ij} for each subsystem (which ensures, that the subsystems are ISS), and seek for conditions, guaranteeing, that the small-gain condition (4.2.11) holds.

Since $\tilde{f}_i \in \mathcal{K}_\infty$ and for all $x > 0$ $c_{ii}(x) < 0$ and $c_{ij}(x) \geq 0,\ i \neq j$, we have that if $x(0) \geq 0$ (that is $x_i(0) \geq 0$ for all $i = 1, \dots, n$) and $u(t) \geq 0$, for all $t > 0$, then $x(t) \geq 0$ for all $t > 0$. Thus, $\mathbb{R}^n_+ = [0, \infty)^n$ is invariant w.r.t. the flow, provided the external inputs are positive.

Pick $V_i(x_i) = |x_i|$ as an ISS-Lyapunov function for i-th entity. Evidently, $V_i(x_i)$ satisfies the condition (4.2.1).

To prove that the condition (4.2.3) holds, we choose the functions γ_{ij}, γ_i, (see (4.2.3)) as

$$\gamma_{ij}(s) := \tilde{f}_i^{-1}\left(\tfrac{a_i}{a_j}\tfrac{1}{1+\delta_j}\tilde{f}_j(s)\right),\ \gamma_i(s) := \tilde{f}_i^{-1}\left(\tfrac{1}{r_i}s\right), \tag{4.2.24}$$

where δ_j, a_j, $j = 1, \dots, n$ and r_i are positive reals. It follows from (4.2.24) that

$$x_i \geq \gamma_{ij}(x_j) \ \Rightarrow \tilde{f}_j(x_j) \leq \tfrac{a_j}{a_i}(1+\delta_j)\tilde{f}_i(x_i),$$

$$x_i \geq \gamma_i(|u_i|) \ \Rightarrow |u_i| \leq r_i\tilde{f}_i(x_i).$$

Using the inequalities from the right hand side of the implications above and assuming that the following condition holds

$$\sum_{j=1, j\neq i}^{n} c_{ij}(x)\tfrac{a_j}{a_i}(1+\delta_j) + c_{ii}(x) + r_i \leq -h_i,\ \forall x \in \mathbb{R}^n_+,\ \textit{for some } h_i > 0, \tag{4.2.25}$$

we obtain that for all $x_i \in \mathbb{R}_+ : V_i(x_i) \geq \max\left\{\max_{j\neq i}\gamma_{ij}\left(V_j(x_j)\right), \gamma_i(|u_i|)\right\}$ it holds that

$$\begin{aligned}\frac{dV_i(x_i(t))}{dt} &= \sum_{j=1}^{n} c_{ij}(x(t))\tilde{f}_j(x_j(t)) + u_i(t)\\ &\leq \left(\sum_{j=1, j\neq i}^{n} c_{ij}(x(t))\tfrac{a_j}{a_i}(1+\delta_j) + c_{ii}(x(t)) + r_i\right)\tilde{f}_i(x_i(t)) \leq -\mu_i(V_i(x_i(t))),\end{aligned}$$

where $\mu_i(r) := h_i\tilde{f}_i(r)$ and thereby condition (4.2.3) is satisfied. Thus, under the condition (4.2.25), $V_i(x_i) = |x_i|$ is an ISS Lyapunov function for the i-th entity with the gains, given by (4.2.24).

To check whether the interconnected system (4.2.23) is ISS we need to verify the small-gain condition (we will use cyclic formulation, see Proposition 4.2.3).

Consider a composition $\gamma_{k_1k_2} \circ \gamma_{k_2k_3}$. It holds

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} = \tilde{f}_{k_1}^{-1}\left(\tfrac{a_{k_1}}{a_{k_2}}\tfrac{1}{1+\delta_{k_3}}\tilde{f}_{k_2}\left(\tilde{f}_{k_2}^{-1}\left(\tfrac{a_{k_2}}{a_{k_3}}\tfrac{1}{1+\delta_{k_3}}\tilde{f}_{k_3}(s)\right)\right)\right) = \tilde{f}_{k_1}^{-1}\left(\tfrac{a_{k_1}}{a_{k_3}}\tfrac{1}{(1+\delta_{k_3})(1+\delta_{k_2})}\tilde{f}_{k_3}(s)\right).$$

In the same way we obtain the expression for the cycle condition in (4.2.12) (here we use that $k_1 = k_p$):

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ \dots \circ \gamma_{k_{p-1}k_p}(s) = \tilde{f}_{k_1}^{-1}\left(\tfrac{1}{\prod_{i=2}^{p}(1+\delta_{k_i})}\tilde{f}_{k_1}(s)\right) < s.$$

Thus, the small gain condition (4.2.12) holds for all $\delta_i > 0$ and by Theorem 4.2.2. the whole system is ISS.

If we assume that the c_{ij} are bounded, i.e., there exists $M > 0$ such that for all $x \in \mathbb{R}^n_+$: $c_{ij}(x) \leq M$ for all $i, j = 1, \dots, n,\ i \neq j$, then the inequality (4.2.25) can be simplified. To this end note that

$$\forall w_i > 0\ \exists \delta_j > 0, j = 1, \dots, n : \sum_{j=1, j\neq i}^{n} c_{ij}(x)\tfrac{a_j}{a_i}\delta_j \leq M\left(\sum_{j=1, j\neq i}^{n}\tfrac{a_j}{a_i}\delta_j\right) < w_i.$$

Using these estimates, we can rewrite (4.2.25) as

$$\sum_{j=1, j\neq i}^{n} c_{ij}(x)a_j \leq -c_{ii}(x)a_i - \epsilon_i,$$

where $\epsilon_i = a_i(r_i + h_i + w_i)$*. In matrix notation, with* $a = (a_1, \ldots, a_n)^T$*,* $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T$*, it takes the form*

$$C(x)a < -\epsilon. \tag{4.2.26}$$

We summarize our investigations in the following proposition.

Proposition 4.2.6. *Consider a network as in* (4.2.22) *and assume that the* c_{ij} *are bounded for all* $i, j = 1, \ldots, n,\ i \neq j$*. If there exist* $a \in \mathbb{R}^n,\ \epsilon \in \mathbb{R}^n,\ a_i > 0,\ \epsilon_i > 0,\ i = 1, \ldots, n$ *such that the condition* $C(x)a < -\epsilon$ *holds for all* $t > 0$*, then the whole network (4.2.23) is ISS.*

Remark 4.2.1. *If* C *is a constant matrix, then the condition* $Ca < -\epsilon$ *is equivalent to* $Ca < 0$ *(with* $a,\ \epsilon$ *as in the proposition above).*

Remark 4.2.2. *Recall, that for the case, when* C *is a constant matrix with negative elements on the main diagonal and all other elements are nonnegative,* C *is diagonal dominant (see, e.g., [10]), if it holds* $c_{ii} + \sum_{j\neq i} c_{ij} < 0$ *for all* $i = 1, \ldots, n$*. In this case, one can easily prove with help of Gershgorin circle theorem (see [10], Fact 4.10.17), that* C *is Hurwitz. Similarly, the previous condition can be replaced with another one: there are* n *numbers* $a_i > 0$ *such that* $c_{ii}a_i + \sum_{j\neq i} c_{ij}a_j < 0$ *for all* $i = 1, \ldots, n$ *(which is equivalent to the existence of a positive vector* a *such that* $Ca < 0$*). In this case the matrix is also Hurwitz (see, e.g., [53]). This shows that Proposition 4.2.6 is consistent with the fact, that linear systems are ISS if and only if matrix* C *is Hurwitz.*

4.3 Constructions of Omega–path

Now we know that existence of an Ω-path makes possible a construction of a Lyapunov function for interconnections of ISS systems. In this section we prove that small-gain condition implies existence of an Ω-path. Moreover, it is possible to give an explicit construction of a 'weak' Ω-path.

Theorem 4.3.1. *If the small-gain condition* (4.2.11) *holds, then there exists an* Ω*-path for* Γ*.*

Proof. For the proof consult [20, Theorem 5.2, (iii)]. □

In [43], Proposition 2.7 and Remark 2.8 a method giving an explicit construction of the 'almost' Ω-path has been developed. Next we describe this method.

Definition 4.3.1. *A function* $\sigma = (\sigma_1, \ldots, \sigma_n)^T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$*, where* $\sigma_i \in \mathcal{K}_\infty$*,* $i = 1, \ldots, n$ *is called a weak* Ω*-path (with respect to operator* Γ*), if it possesses the following properties:*

1. σ_i^{-1} *is locally Lipschitz continuous on* $(0, \infty)$*;*
2. *for every compact set* $P \subset (0, \infty)$ *there are finite constants* $0 < K_1 < K_2$ *such that for all points of differentiability of* σ_i^{-1} *we have*

$$0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2, \quad \forall r \in P;$$

3. $$\Gamma(\sigma(r)) \leq \sigma(r),\ \forall r > 0. \tag{4.3.1}$$

As we see the difference of the weak Ω-path from the usual Ω-path is only in the inequality (4.3.1).

Definition 4.3.2. *For arbitrary $h_i = (h_{1i}, \dots, h_{ni})^T \in \mathbb{R}^n$, $i = 1, \dots, m$ define $MAX(h_1, \dots, h_m) := z \in \mathbb{R}^n$ with $z_i := \max\{h_{1i}, \dots, h_{mi}\}$.*

Remark 4.3.1. *It is easy to check that for any $x, y, z \in \mathbb{R}^n$ holds*

$$MAX(x, MAX(y, z)) = MAX(x, y, z). \tag{4.3.2}$$

Definition 4.3.3. *$G : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is called nondecreasing if $G(x) \geq G(y)$ for arbitrary $x, y \in \mathbb{R}^n_+$ satisfying $x \geq y$.*

Definition 4.3.4. *We say that $G : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is MAX-preserving if G is nondecreasing and for every $x, y \in \mathbb{R}^n_+$ the following equality holds:*

$$G(MAX(x, y)) = MAX(G(x), G(y)) \tag{4.3.3}$$

Proposition 4.3.2. *$G : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ with $G(x) = (G_1(x), \dots, G_n(x))^T$ is MAX-preserving if and only if there exist non-decreasing functions $\gamma_{i,j} : \mathbb{R}_+ \to \mathbb{R}_+$, $i, j = 1, \dots, n$ with $G_i(x) = \max_{j=1,\dots,n} \gamma_{i,j}(x_j)$, for all $x \in \mathbb{R}^n_+$, $i = 1, \dots, n$.*

Proof. See [43, Proposition 2.6]. □

Now let $G^n(x) = G \circ G^{n-1}(x)$, for all $n \geq 2$. Define map $Q_G : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

$$Q_G(x) := MAX(x, G(x), G^2(x), \dots, G^{n-1}(x)).$$

As a consequence of Proposition 4.3.2, Proposition 4.2.3 and [43, Proposition 2.7] we see that

Proposition 4.3.3. *$\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, defined by (4.2.4) is a MAX-preserving operator. In addition, if Γ satisfies small-gain condition (4.2.11), then for all $x \in \mathbb{R}^n_+$ and all $k \in \mathbb{N}$ the following inequality holds*

$$\Gamma^k(x) \leq Q_\Gamma(x). \tag{4.3.4}$$

Proposition 4.3.3 gives a method to construct a 'weak'-Ω-path for arbitrary gains.

By $int(S)$ we denote the interior of $S \subset \mathbb{R}^n$.

Proposition 4.3.4. *Let operator Γ satisfy small-gain condition. Pick any $a \in int(\mathbb{R}^n_+)$. Then σ, defined by*

$$\sigma(t) = Q_\Gamma(at) \quad \forall t \geq 0 \tag{4.3.5}$$

satisfies (4.3.1).

Proof. We have

$$\begin{aligned} \Gamma(\sigma(t)) &= \Gamma(Q_\Gamma(at)) \\ &= \Gamma \circ MAX(at, \Gamma(at), \Gamma^2(at), \dots, \Gamma^{n-1}(at)). \end{aligned}$$

Since Γ is MAX-preserving, we can continue estimates to obtain

$$\Gamma(\sigma(t)) = MAX(\Gamma(at), \Gamma^2(at), \Gamma^3(at), \dots, \Gamma^n(at)).$$

Now, using the property (4.3.4) and then the property (4.3.2) we obtain

$$\begin{aligned} \Gamma(\sigma(t)) &\leq MAX(\Gamma(at), \Gamma^2(at), \Gamma^3(at), \dots, Q_\Gamma(at)) \\ &\leq MAX(at, \Gamma(at), \Gamma^2(at), \Gamma^3(at), \dots, \Gamma^{n-1}(at)) \\ &= Q_\Gamma(at) \\ &= \sigma(t). \end{aligned}$$

□

If the gains γ_{ij} and their inverse γ_{ij}^{-1} are smooth enough, then σ, constructed in Proposition 4.3.4 is a weak Ω-path w.r.t. Γ.

4.4 Interconnections of linear systems

The construction of ISS-Lyapunov function for the interconnections of finite-dimensional input-to-state stable linear systems (see [20]) can be generalized to the case of interconnections of linear systems over Banach spaces.

Consider the following interconnected system

$$\dot{x}_i = A_i x_i(t) + \sum_{j=1}^{n} B_{ij} x_j(t) + C_i u(t),\ i = 1, \dots, n, \tag{4.4.1}$$

where $x_i(t) \in \mathbb{R}^{p_i}$, $A_i \in \mathbb{R}^{p_i \times p_i}$, $i = 1, \dots, n$, $B_{ij} \in \mathbb{R}^{p_i \times p_j}$, $i, j \in \{1, \dots, n\}$ and $u \in \mathcal{U}$. We assume, that $B_{ii} = 0$, $i = 1, \dots, n$. Otherwise we can always substitute $\tilde{A}_i = A_i + B_{ii}$.

We introduce the matrices $A := diag(A_1, \dots, A_n) \in \mathbb{R}^{n \times n}$, $B := (B_{ij})_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}$ and $C := (C_1, \dots, C_n)^T$, a column matrix, consisting of C_i. Then the system (4.4.1) can be rewritten in the following form

$$\dot{x}(t) = (A + B)x(t) + Cu(t). \tag{4.4.2}$$

We are going to apply Lyapunov technique developed in this section to the system (4.4.1).

According to Proposition 3.1.2 the i-th subsystem of (4.4.1) is ISS iff it is GAS. This implies due to Theorem 2.4.3 that there exists a positive operator P_i so that

$$A_i^T P_i + P_i A_i = -I \tag{4.4.3}$$

holds. Moreover, the function

$$V_i(x_i) = x_i^T P_i x_i \tag{4.4.4}$$

is a GAS Lyapunov function for the i-th subsystem of (4.4.1).

Note that since P_i is a positive matrix, there exist certain $a_i > 0$ so that:

$$a_i^2 |x_i|^2 \leq V_i(x_i) \leq \|P_i\| |x_i|^2. \tag{4.4.5}$$

Differentiating V_i w.r.t. the i-th subsystem of (4.4.1), we obtain for all $x_i \in \mathbb{R}^{p_i}$

$$\begin{aligned} \dot{V}_i(x_i) &= \dot{x}_i^T P_i x_i + x_i^T P_i \dot{x}_i \\ &\leq (A_i x_i)^T P_i x_i + x_i^T P_i A_i x_i + 2|x_i| \|P_i\| \left(\sum_{i \neq j} \|B_{ij}\| |x_j| + \|C_i\| |u| \right). \end{aligned}$$

Using (4.4.3), we obtain

$$\dot{V}_i(x_i) \leq -|x_i|^2 + 2|x_i| \|P_i\| \left(\sum_{i \neq j} \|B_{ij}\| |x_j| + \|C_i\| |u| \right).$$

Now take $\varepsilon \in (0, 1)$ and let

$$|x_i| \geq \frac{2\|P_i\|}{1 - \varepsilon} \left(\sum_{i \neq j} \|B_{ij}\| |x_j| + \|C_i\| |u| \right). \tag{4.4.6}$$

Then we obtain for all $x_i \in \mathbb{R}^{p_i}$

$$\dot{V}_i(x_i) \leq -\varepsilon |x_i|^2.$$

In order to apply small-gain theorem, we replace inequality (4.4.6) by the following one

$$V_i(x_i) \geq \|P_i\|^3 \left(\frac{2}{1-\varepsilon}\right)^2 \left(\sum_{i\neq j} \frac{\|B_{ij}\|}{a_j}\sqrt{V_j(x_j)} + \|C_i\|\|u\|_U\right)^2. \tag{4.4.7}$$

It is easy to see that (4.4.7) together with (4.4.5) imply (4.4.6).

Thus, gains can be defined by:

$$\gamma_{ij}(s) = \left(\frac{2\|P_i\|^{3/2}}{1-\varepsilon}\frac{\|B_{ij}\|}{a_j}\right)\sqrt{s}, \tag{4.4.8}$$

for all $i \neq j$, $i = 1,\dots,n$. If the small-gain condition for this choice of gains holds, we can conclude the ISS of the system (4.4.2).

4.5 Tightness of small-gain conditions

Theorem 4.6.1 states, that if all the subsystems are ISS in summation formulation then small-gain condition (4.6.1) is sufficient for input-to-state stability of the whole system. However, the small-gain condition is not necessary for ISS of an interconnection and the question arises, how tight it is. A partial answer is given by the following theorem

Theorem 4.5.1. *Let a gain matrix $\Gamma := (\gamma_{ij}),\ i,j = 1,\dots,n,\ \gamma_{ii} = 0$ be given. If the condition (4.2.11) is not satisfied, then there exists a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ so that $\forall i = 1,\dots,n$ estimates (4.1.5) hold $\forall t \geq 0$, but the whole system (4.1.1) is not 0-GAS.*

Proof. For arbitrary gain matrix Γ, satisfying the assumptions of the theorem, we are going to construct a corresponding system satisfying (4.1.5), but which is not 0-GAS.

Let Γ does not satisfy (4.2.11). According to Proposition 4.2.3, there exists some cycle such that the condition (4.2.12) is violated. Let $\exists s > 0$, such that

$$\gamma_{12} \circ \gamma_{23} \circ \ldots \circ \gamma_{r-1r} \circ \gamma_{r1}(s) \geq s, \tag{4.5.1}$$

where $2 \leq r \leq n$ (violation of the small-gain condition on another cycles can be treated in the same way).

Due to continuity of γ_{ij}, there exist constants $\varepsilon_i \in [0,1)$, $i = 2,\dots,r$, such that for functions $\chi_{ij} := (1-\varepsilon_j)\gamma_{ij}$ and the same s it holds that

$$\chi_{12} \circ \chi_{23} \circ \ldots \circ \chi_{r-1r} \circ \chi_{r1}(s) = s. \tag{4.5.2}$$

Let us enlarge the domain of definition of functions χ_{ij} to $\mathbb{R}$, defining $\chi_{ij}(-p) = -\chi_{ij}(p)\ \forall p > 0,\ i,j = 1,\dots n,\ i \neq j$.

Consider the following system:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + \chi_{12}\left(x_2(t)\right) \\ \dot{x}_2(t) = -x_2(t) + \chi_{23}\left(x_3(t)\right) \\ \dots \\ \dot{x}_r(t) = -x_r(t) + \chi_{r1}\left(x_1(t)\right) \\ \dot{x}_{r+1}(t) = -x_{r+1}(t) \\ \dots \\ \dot{x}_n(t) = -x_n(t) \end{cases} \tag{4.5.3}$$

For the first equation, using variation of constants formula, we obtain the following estimates:

$$\begin{aligned}
|x_1(t)| &\leq |x_1(0)|\, e^{-t} + \left| \int_0^t e^{s-t} \chi_{12}(x_2(s)) ds \right| \\
&\leq |x_1(0)|\, e^{-t} + e^{-t} \int_0^t e^{s} \left| \chi_{12}(x_2(s)) \right| ds \\
&= |x_1(0)|\, e^{-t} + e^{-t} \int_0^t e^{s} \chi_{12}(|x_2(s)|) ds \\
&\leq |x_1(0)|\, e^{-t} + e^{-t} \int_0^t e^{s} ds \chi_{12}\left(\|x_2\|_\infty \right) \\
&\leq |x_1(0)|\, e^{-t} + \chi_{12}\left(\|x_2\|_\infty \right).
\end{aligned}$$

Similar estimates can be made for all equations. Thus, inequalities (4.1.5) are satisfied. Now we are going to prove, that the system (4.5.3) is not 0-GAS.

Fixed points of the system (4.5.3) are the solutions $(x_1, \ldots, x_n)$ of the following system:

$$\begin{cases}
x_1 = \chi_{12}\left(x_2\right) \\
x_2 = \chi_{23}\left(x_3\right) \\
\ldots \\
x_{r-1} = \chi_{r-1r}\left(x_r\right) \\
x_r = \chi_{r1}\left(x_1\right) \\
x_i = 0, \;\; i = r+1, \ldots, n
\end{cases} \tag{4.5.4}$$

Substituting the i-th equation of (4.5.4) into the $(i-1)$-th, $i = r, \ldots, 2$, we obtain the equivalent system:

$$\begin{cases}
x_1 = \chi_{12} \circ \chi_{23} \circ \ldots \circ \chi_{r1}\left(x_1\right) \\
x_2 = \chi_{23} \circ \chi_{34} \circ \ldots \circ \chi_{r1}\left(x_1\right) \\
\ldots \\
x_{r-1} = \chi_{r-1r} \circ \chi_{r1}\left(x_1\right) \\
x_r = \chi_{r1}\left(x_1\right) \\
x_i = 0, \;\; i = r+1, \ldots, n
\end{cases} \tag{4.5.5}$$

For all solutions $s > 0$ of the equation (4.5.2), the first equation of the system (4.5.5) is satisfied with $x_1 = s$, and a point

$$(x_1, \ldots, x_{r-1}, x_r, \ldots, x_n) = (s, \chi_{23} \circ \chi_{34} \circ \ldots \circ \chi_{r-1r} \circ \chi_{r1}(s), \ldots, \chi_{r-1r} \circ \chi_{r1}\left(s\right), \chi_{r1}(s), 0, \ldots, 0)$$

is a fixed point for the system (4.5.3). Hence the system (4.5.3) has a nonzero fixed point and therefore it is not 0-GAS. □

The counterpart of this result can be proved also for the Lyapunov-type small gain theorem.

Theorem 4.5.2. *Let a matrix of Lyapunov gains $\Gamma := (\gamma_{ij}),\ i,j = 1, \ldots, n,\ \gamma_{ii} = 0$ be given. Let there exist $s > 0$, such that for some cycle in Γ it holds*

$$\gamma_{12} \circ \gamma_{23} \circ \ldots \circ \gamma_{r-1r} \circ \gamma_{r1}(s) > s, \tag{4.5.6}$$

where $2 \leq r \leq n$ (we can always renumber the nodes to obtain the cycle of the needed form). Then there exist a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and Lyapunov functions V_i for subsystems (in maximum formulation), so that $\forall i = 1, \ldots, n$ it holds (4.2.3), but the whole system (4.1.1) is not 0-GAS.

Proof. Take constants $\varepsilon_i \in (0,1)$, $i = 2,\dots,r$, such that for the functions $\chi_{ij} := (1-\varepsilon_i)\gamma_{ij}$ and some $s > 0$ it holds

$$\chi_{12} \circ \chi_{23} \circ \ldots \circ \chi_{r-1r} \circ \chi_{r1}(s) = s.$$

Consider the system (4.5.3). Take $V_i(x_i) = |x_i|$ as Lyapunov functions for i-th subsystem. For $i = 1,\dots,r-1$ if

$$V_i(x_i) \geq \gamma_{ii+1}(x_{i+1}) = \frac{1}{1-\varepsilon_i}\chi_{ii+1}(x_{i+1})$$

holds, then

$$\dot{V}_i(x_i) \leq -V_i(x_i) + (1-\varepsilon_i)V_i(x_i) = -\varepsilon_i V_i(x_i).$$

Thus, for $i = 1,\dots,r-1$ V_i is an ISS Lyapunov function for i-th subsystem. In fact, it holds also for all $i = 1,\dots,n$. Moreover, Γ is a matrix of Lyapunov gains for the system (4.5.3). According to the proof of the Theorem 4.5.1, (4.5.3) is not a 0-GAS system. □

We discuss the obtained results in the end of the next section.

4.6 Concluding remarks

ISS framework is not the only existing tool to study the interconnections of the dynamical systems. In particular, small-gain theorems were originally established within input-output approach to stability of control systems, see [48, Chapter 5].

Another framework is a dissipative systems theory, originated from papers [82], [83] by J. Willems. An important theorem in this framework is that a feedback interconnection of dissipative systems is again dissipative. An important special case of dissipative systems are passive systems [78]. Closely connected to passive systems are port-Hamiltonian systems, widely used in modeling and analysis of finite and infinite-dimensional control systems [23].

The study of interconnections of control systems plays an important role in behavioral approach [84], [65] to dynamical systems theory. The small-gain theorems arise also within this framework, see e.g. [12].

We have proved only two basic small-gain theorems. Many generalizations and versions of these results are available in the literature: [43], [41], [20].

For an introduction to ISS theory with bias to the stability of interconnected systems see [31, Chapter 10].

In [22] an ISS small gain theorem for networks in terms of trajectories was proved, namely

Theorem 4.6.1. *Let all subsystems of system (4.1.1) be ISS in maximum formulation. If the corresponding gain operator satisfies the small gain condition (4.2.11) then the whole system (4.1.1) is ISS.*

For a summation formulation the same statement holds, but with a stronger small-gain condition:

$$D \circ \Gamma(s) \not\geq s,\ \forall\ s \in \mathbb{R}^n_+ \setminus \{0\}\,, \tag{4.6.1}$$

for some $D = diag(id + \alpha_1, \dots, id + \alpha_n)$, $\alpha_i \in \mathcal{K}_\infty$.

This theorem is a generalization of the small-gain theorem for an interconnection of two systems, proved in [39].

Chapter 5

Input-to-state stabilization

One of the central problems in control theory is a stabilization of nonlinear systems. Since every control system is subject to external disturbances, which may be due errors arising in computing of a controller, unmodelled dynamics etc, the stabilization should be achieved in a robust way. In this section we show that ISS plays a foundational role in the modern stabilization theory.

Consider a system

$$\dot{x} = f(x, u, d), \ t > 0 \tag{5.0.1}$$

where $x(t) \in \mathbb{R}^n$, $u \in \mathcal{U}$, and $d \in \mathcal{D} := L_\infty(\mathbb{R}_+, \mathbb{R}^p)$ for some $n, p \in \mathbb{N}$.

This system possesses two inputs:

- u is a control, which we can choose in order to achieve certain aims.
- d which we consider as a disturbance, which we cannot influence.

Definition 5.0.1. (5.0.1) *is called ISS stabilizable if there exists a feedback $k : \mathbb{R}^n \to \mathbb{R}^m$, so that*

$$\dot{x}(t) = f(x(t), k(x(t)), d(t)), \ t > 0 \tag{5.0.2}$$

is ISS w.r.t. the disturbance d. The feedback k is called an ISS stabilizing control (feedback).

Definition 5.0.2. (5.0.1) *is called GAS stabilizable if there exists a feedback $k : \mathbb{R}^n \to \mathbb{R}^m$, so that*

$$\dot{x}(t) = f(x(t), k(x(t)), 0), \ t > 0$$

is GAS. The feedback k is called a stabilizing control (feedback).

In what follows we develop several basic methods of robust stabilization of control systems.

5.1 ISS feedback redesign

Consider a special case of the system (5.0.1) where disturbances enter the system in a special way, namely, a disturbance d distorts the control signal sent to the system, but does not change the structure of the system:

$$\dot{x}(t) = f(x(t), u(t) + d(t)) \tag{5.1.1}$$

Such disturbances are called the actuator disturbances. The schematic description for such a system in absence and in presence of disturbances is depicted in Figures 5.1, 5.2.

Let us consider a stabilization problem for the following simple system:

$$\dot{x} = x + (x^2 + 1)(u + d). \tag{5.1.2}$$

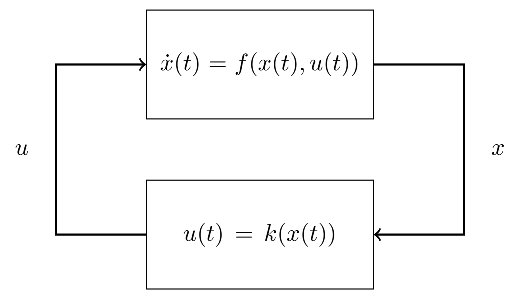

Figure 5.1: The stabilization problem without actuator disturbances

First assume that $d \equiv 0$.

Pick $k(x) := -\frac{2x}{1+x^2}$. The feedback control $u(t) := k(x(t))$ globally (asymptotically) stabilizes the system (5.1.2) with $d \equiv 0$, and the closed-loop system takes form:

$$\dot{x} = -x. \tag{5.1.3}$$

The stabilization problem for the undisturbed system (5.1.2) is solved.

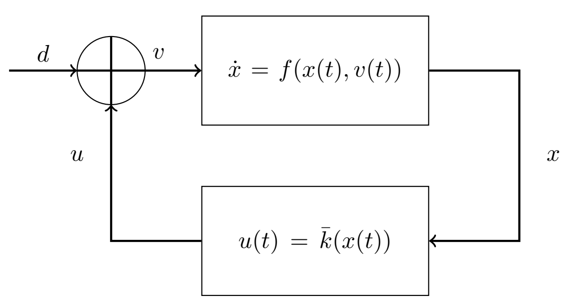

Figure 5.2: The stabilization problem with actuator disturbances

Now assume that actuator disturbances are present. Using the same stabilizing controller, we obtain after substitution of $u := k(x)$ into (5.1.2) the following equation

$$\dot{x} = -x + (x^2 + 1)d.$$

This system is 0-GAS, but for disturbance $d \equiv 1$ the state of the system blows up in finite time for any initial condition. We see that a feedback controller which stabilizes an undisturbed system may have a poor performance in presence of actuator disturbances. This motivates a natural question, whether it is possible to modify the stabilizing feedback in a way to make the controller ISS stabilizing the system in presence of disturbances. Next we positively answer this question for the special class of control-affine systems

$$\dot{x}(t) = g_0(x) + g_1(x)(u + d), \tag{5.1.4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$.

We have the following result:

Theorem 5.1.1. *Let (5.1.4) be GAS stabilizable for $d \equiv 0$. Then (5.1.4) is ISS stabilizable possibly by means of another stabilizing controller.*

Proof. Let (5.1.4) be GAS stabilizable for $d \equiv 0$. Then there exists $k = k(x(t))$ so that

$$\dot{x}(t) = g_0(x) + g_1(x)k(x(t)), \tag{5.1.5}$$

is GAS.

Due to the converse Lyapunov theorem (Theorem 2.5.1) there exists a smooth Lyapunov function V for (5.1.5), satisfying for some $\alpha \in \mathcal{K}_\infty$ and all $x \neq 0$

$$\dot{V}_{d=0}(x) = (\nabla V(x))^T \Big(g_0(x) + g_1(x)k(x(t))\Big) \leq -\alpha(x(t)). \tag{5.1.6}$$

Let us design a controller $\bar{k} : \mathbb{R}^n \to \mathbb{R}$, which ISS stabilizes (5.1.4). The Lie derivative of V (which we denote $\dot{V}_d(x)$) w.r.t. the system

$$\dot{x}(t) = g_0(x) + g_1(x)\big(\bar{k}(x(t)) + d\big).$$

We have:

$$\begin{aligned}\dot{V}_d(x) &= (\nabla V(x))^T \Big(g_0(x) + g_1(x)\big(\bar{k}(x(t)) + d\big)\Big) \\ &= (\nabla V(x))^T \Big(g_0(x) + g_1(x)k(x(t))\Big) + (\nabla V(x))^T g_1(x)\Big(\bar{k}(x(t)) - k(x(t)) + d\Big)\end{aligned}$$

Choose

$$\bar{k}(x(t)) := k(x(t)) - (\nabla V(x))^T g_1(x).$$

We continue estimates using this $\bar{k}$ and inequality (5.1.6) to obtain

$$\begin{aligned}\dot{V}_d(x) &\leq -\alpha(x(t)) - \Big((\nabla V(x))^T g_1(x)\Big)^2 + (\nabla V(x))^T g_1(x) d \\ &\leq -\alpha(x(t)) - \Big((\nabla V(x))^T g_1(x)\Big)^2 + \frac{1}{2}\Big((\nabla V(x))^T g_1(x)\Big)^2 + \frac{1}{2}|d|^2 \\ &= -\alpha(x(t)) - \frac{1}{2}\Big((\nabla V(x))^T g_1(x)\Big)^2 + \frac{1}{2}|d|^2.\end{aligned}$$

This shows that V is an ISS Lyapunov function in a dissipative form and $\bar{k}$ is an ISS stabilizing controller for (5.1.4). □

Let us apply Theorem 5.1.1 for the system (5.1.3).

The GAS Lyapunov function for (5.1.3) can be chosen as $V(x) := \frac{1}{2}x^2$. According to Theorem 5.1.1 the following controller

$$\bar{k}(x(t)) = -\frac{2x}{1+x^2} - x(1+x^2). \tag{5.1.7}$$

ISS stabilizes (5.1.2).

Exercise 5.1.1. *Theorem 5.1.1 does not always produce the simplest controllers. Show that*

$$u(t) := \bar{k}_2(x(t)) = -\frac{2x(t) + x^3(t)}{1 + x^2(t)}$$

also ISS stabilizes the system (5.1.2).

Exercise 5.1.2. *Prove a result, similar to Theorem 5.1.1 for the more general input-affine system*

$$\dot{x}(t) = g_0(x) + \sum_{i=1}^{m} g_i(x) u_i,$$

where $g_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, \ldots, m$ and $u_i(t) \in \mathbb{R}$, for all $i = 1, \ldots, m$.

5.2 Applications of ISS to design of robust nonlinear observers

In Section 5.1 we investigated a problem of an ISS stabilization of nonlinear systems (5.0.1) by means of a static feedback $u(t) = k(x(t))$. However, in many cases it is not possible to measure the whole state of a system, e.g. due to high costs of sensors or due to technical impossibility of such measurements. Instead, we can measure only a certain function of the state $y = h(x)$ which is called an output of the system. In such a case we are not able to construct the stabilizing feedback as a function of x, and another ways of stabilization should be investigated.

The system (5.0.1) together with the output law forms a *control system with outputs*

$$\begin{aligned} \dot{x} &= f(x, u, d), \\ y &= h(x), \end{aligned} \tag{5.2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$.

The stabilization of (5.2.1) by means of an output feedback $u(t) = k(y(t))$ may be impossible even for linear systems which are controllable and observable, as the next example shows

Example 5.2.1. *Consider the double integrator system*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u \end{aligned} \tag{5.2.2}$$

with the output $y = x_1$. This system can be rewritten in a matrix form as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x. \tag{5.2.3}$$

We are going to show that (5.2.3) is not stabilizable by means of a continuous static output feedback. Assume that there exist a continuous function $k : \mathbb{R} \to \mathbb{R}$, so that the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= k(x_1) \end{aligned} \tag{5.2.4}$$

is asymptotically stable. Consider the function $V(x) = x_2^2 - 2\int_0^{x_1} k(a)\, da$. It holds that

$$\frac{d}{dt} V(\varphi(t)) = 2x_2 k(x_1) - 2k(x_1)x_2 = 0.$$

Due to definition of V we have $V((0,0)) = 0$ and $V((0,1)) = 1$. Since V is continuous, it follows that $V(\varphi(t,(0,1))) \equiv 1$. Thus, $\varphi(t,(0,1))$ does not converge to 0 and hence (5.2.3) is not stabilizable by means of a continuous static output feedback.

The drawbacks of a static output feedback lead to the problem how to reconstruct the state of the system (5.2.1) provided that only the information about the input and output is available.

Definition 5.2.1. *A robust observer of a nonlinear system (5.2.1) is a system of the form*

$$\dot{\bar{x}} = g(\bar{x}, y, u), \quad \bar{x}(t) \in \mathbb{R}^{n_0} \tag{5.2.5}$$

for which there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ so that the error $e(t, x, \bar{x}, u, d) := \phi(t, x, u, d) - \bar{\phi}(t, \bar{x}, u, y)$ between the states of (5.2.1) and (5.2.5) satisfies

$$|e(t, x, \bar{x}, u, d)| \leq \beta(|x - \bar{x}|, t) + \gamma(\|d\|_\infty)$$

for all $x \in \mathbb{R}^n$, $\bar{x} \in \mathbb{R}^{n_0}$, for any input $u \in \mathcal{U}$ and for all $t \geq 0$.

According to the above definition the state of an observer converges to the state of a modeled nonlinear system when $t \to \infty$. Graphically an observer can be described as in Figure 5.3.

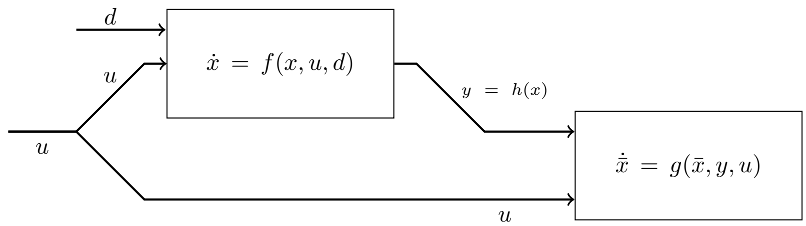

Figure 5.3: Definition of an observer

5.2.1 Luenberger's observer and dynamic feedback

In this section we present a well-known design of observers for linear systems due to Luenberger. Consider a linear control system

$$\dot{x} = Ax + Bu + Dd, \tag{5.2.6}$$
$$y = Cx, \tag{5.2.7}$$

where $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, where $n, p \in \mathbb{N}$.

We have the following result, which dates back to Luenberger

Theorem 5.2.2 (Luenberger's Observers). *Let there exists L so that $A + LC$ is Hurwitz. Then the following system is an observer for* (5.2.7)*:*

$$\dot{\bar{x}} = A\bar{x} + L(C\bar{x} - y) + Bu \tag{5.2.8}$$

Proof. Let L be so that $A + LC$ is Hurwitz. Consider the Luenberger's observer (5.2.8) and define the error between exact and approximate states by $e := x - \bar{x}$. Subtracting the equation (5.2.8) from (5.2.7) we get the following system for the error e:

$$\dot{e} = Ae - L(C\bar{x} - Cx) + Dd$$

which can be rewritten as

$$\dot{e} = (A + LC)e + Dd. \tag{5.2.9}$$

Since $A + LC$ is Hurwitz, (5.2.9) is 0-GAS and due to Proposition 3.1.2 (5.2.9) is ISS. This shows that (5.2.8) in a robust observer for (5.2.7). □

Next we show that difficulties encountered in Example 5.2.1 can be overcome by means of the dynamic feedback.

Theorem 5.2.3. *Consider* (5.2.7)*. Let there exist F, L so that $A+BF$ and $A+LC$ are Hurwitz matrices. Then* (5.2.7) *is stabilizable by means of a feedback $u = F\bar{x}$, where $\bar{x}$ comes from* (5.2.8)*.*

Proof. Consider the system (5.2.7) coupled with the observer (5.2.8) by means of $u = F\bar{x}$. We obtain a coupled system

$$\begin{aligned} \dot{x} &= Ax + BF\bar{x} + Dd, \\ \dot{\bar{x}} &= A\bar{x} + L(C\bar{x} - Cx) + BF\bar{x} \end{aligned} \tag{5.2.10}$$

which can be rewritten as

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{\bar{x}}\end{pmatrix} &= \begin{pmatrix}A & BF\\ -LC & A+LC+BF\end{pmatrix}\begin{pmatrix}x\\ \bar{x}\end{pmatrix} + \begin{pmatrix}D\\ 0\end{pmatrix} d\\ &= T^{-1}\begin{pmatrix}A & BF\\ -LC & A+LC+BF\end{pmatrix} T \begin{pmatrix}x\\ \bar{x}\end{pmatrix} + \begin{pmatrix}D\\ 0\end{pmatrix} d.\end{aligned}$$

Here $T = \begin{pmatrix} I & 0\\ -I & I\end{pmatrix}$ and $T^{-1} = \begin{pmatrix} I & 0\\ I & I\end{pmatrix}$.

Since $A+BF$ and $A+LC$ are assumed to be Hurwitz matrices, $\begin{pmatrix} A+BF & BF\\ 0 & A+LC\end{pmatrix}$ as well its equivalent transformation $T^{-1}\begin{pmatrix} A+BF & BF\\ 0 & A+LC\end{pmatrix} T$ are also Hurwitz matrices. This shows that (5.2.10) is 0-GAS and (5.2.10) is ISS due to Proposition 3.1.2. □

Exercise 5.2.1. *Find an observer for the system* (5.2.2) *and stabilize it by means of a dynamic feedback.*

5.2.2 Observers of nonlinear systems

As we have seen, the Luenberger's observer constructed in the previous section is automatically robust due to linearity of the error dynamics. For nonlinear systems robustness of the observer comes to the forefront of research. Next we construct a robust observer for the special class of nonlinear systems. We follow in our analysis [5].

Consider

$$\dot{x} = Ax + G\big(\gamma(Hx) + \Delta(Hx)d(t)\big) + \rho(y,u), \quad (5.2.11)$$

$$y = Cx, \quad (5.2.12)$$

where $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$ for certain $n, p \in \mathbb{N}$ and $\gamma(Hx)$ is an r-dimensional vector with the i-th component of the form

$$\gamma_i = \gamma_i(\sum_{i=1}^{n} H_{ij}x_j).$$

is an r-dimensional vector function.

In (5.2.12) the disturbance d does not distort the input signal (as it is the case in actuator disturbances), but invokes the additional unmodeled dynamics $\Delta(Hx)$ into play.

Consider the following observer candidate for (5.2.12):

$$\dot{\bar{x}} = A\bar{x} + L(C\bar{x} - y) + G\gamma(H\bar{x} + K(C\bar{x} - y)) + \rho(y,u). \quad (5.2.13)$$

We are going to prover that (5.2.13) is indeed a robust observer for (5.2.12) provided certain assumptions are fulfilled.

Theorem 5.2.4. *Assume that for all $i = 1,\dots,r$ there exist $\sigma_i \in \mathcal{K}$ so that for all $a,b,d \in \mathbb{R}$ it holds that*

$$(a-b)(\gamma_i(a) - \gamma_i(b) + \Delta_i(a)d) \geq -\sigma_i(|d|) \quad (5.2.14)$$

Let also there exist a matrix $P = P^T > 0$, diagonal matrix $\Lambda > 0$ and $\nu > 0$ so that

$$\begin{pmatrix}(A+LC)^TP + P(A+LC) + \nu I & PG + (H+KC)^T\Lambda\\ G^TP + \Lambda(H+KC) & 0\end{pmatrix} \leq 0 \quad (5.2.15)$$

holds for certain matrices K, L.

Then (5.2.13) *is a robust observer for the system* (5.2.12).

Proof. Let the assumptions of the theorem hold.

As before, we denote $e(t) := x(t) - \bar{x}(t)$ and consider the system governing the error dynamics

$$\dot{e} = (A + LC)e + G(\gamma(v) - \gamma(w) + \Delta(v)d) \tag{5.2.16}$$

where $v = Hx$ and $w = H\bar{x} + K(C\bar{x} - y)$.

It is of virtue to introduce a new variable $z := v - w$. Note that $z = (H + KC)e$. Also we denote $\varphi(t, z, d) := \gamma(v) - \gamma(w) + \Delta(v)d$.

Note that the condition (5.2.14) for $a := v$ and $b := w$ leads for all $i = 1, \dots, r$ to

$$z_i \varphi_i(t, z_i, d) \geq -\sigma_i(|d|), \quad \forall z_i \in \mathbb{R}, t \geq 0, d \in \mathbb{R}. \tag{5.2.17}$$

Now we can rewrite (5.2.16) as

$$\dot{e} = (A + LC)e + G\varphi(t, z, d) \tag{5.2.18}$$

Consider the following ISS Lyapunov function candidate:

$$V(e) = e^T P e. \tag{5.2.19}$$

Since $P > 0$, V satisfies the condition (2.3.1). Let us compute Lie derivative of V:

$$\begin{aligned} \dot{V}(e) &= \dot{e}^T P e + e^T P \dot{e} \\ &= \big((A + LC)e + G\varphi(t, z, d)\big)^T P e + e^T P \big((A + LC)e + G\varphi(t, z, d)\big) \\ &= e^T \big((A + LC)^T P + P(A + LC)\big)e + \varphi^T(t, z, d) G^T P e + e^T P G \varphi(t, z, d) \\ &= e^T \big((A + LC)^T P + P(A + LC) + \nu I\big)e \qquad - \nu e^T e \\ &\quad + \varphi^T(t, z, d)\big(G^T P + \Lambda(H + KC)\big)e \qquad - \varphi^T(t, z, d)\Lambda(H + KC)e \\ &\quad + e^T\big(PG + (H + KC)^T\big)\varphi(t, z, d) \qquad - e^T(H + KC)^T \varphi(t, z, d). \end{aligned}$$

Applying the inequality (5.2.15) and recalling that $z = (H + KC)e$, we see that

$$\begin{aligned} \dot{V}(e) &\leq -\nu|e|^2 - \varphi^T(t, z, d)\Lambda z - z^T \Lambda \varphi(t, z, d) \\ &= -\nu|e|^2 - 2z^T \Lambda \varphi(t, z, d) \\ &= -\nu|e|^2 - 2\sum_{i=1}^{r} z_i \lambda_i \varphi_i(t, z_i, d). \end{aligned}$$

Since $\Lambda > 0$ and due to (5.2.17) we finally arrive at

$$\dot{V}(e) \leq -\nu|e|^2 + 2\sum_{i=1}^{r} \sigma_i(|d|) \tag{5.2.20}$$

$$\leq -\frac{\nu}{\lambda_{min}(P)} V(e) + \sigma(|d|), \tag{5.2.21}$$

with $\sigma(|d|) := 2\sum_{i=1}^{r} \sigma_i(|d|)$. This shows that V is an exponential ISS Lyapunov function for (5.2.16) and thus (5.2.13) is a robust observer for (5.2.12). □

Chapter 6

Conclusion and outlook

In these notes we have studied ISS of ordinary differential equations. We have seen that ISS unified internal stability as well as stability w.r.t. external inputs and helped to solve a number of important problems in nonlinear control theory. In particular, ISS is a basis for robust stability and stabilizability theory, for design of robust observers, ISS feedback redesign etc. But this does not deplete all the power of ISS theory. The importance of the ISS concept for nonlinear control theory has led to the development of related notions, refining and/or generalizing ISS in some sense: integral ISS [69, 4], input-to-output stability (IOS) [39, 46], input-output-to-state stability (IOSS) [72, 51], input-to-state dynamical stability (ISDS) [25], incremental ISS [30, 2], to mention a few.

Next we describe very briefly these concepts.

Integral input-to-state stability (iISS). ISS does not include all systems with a certain kind of robustness. In particular, ISS excludes systems whose state stays bounded as long as the magnitude of applied inputs remains below a specific threshold, but becomes unbounded when the input magnitude exceeds the threshold. Such behavior is frequently caused by saturation and limitations in actuation and processing rate. The idea of integral input-to-state stability (iISS) is to capture those nonlinearities [69, 4].

Definition 6.0.2. *System is called integral input-to-state stable (iISS) if there exist $\alpha \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that the inequality*

$$\alpha(|\phi(t,x,u)|) \leq \beta(|x|,t) + \int_0^t \mu(|u(s)|)ds \tag{6.0.1}$$

holds $\forall x \in \mathbb{R}^n$, $\forall u \in \mathcal{U}$ and $\forall t \geq 0$.

The notion of iISS Lyapunov function, the Lyapunov characterization of iISS property and characterizations of iISS property have been developed in [69, 4]. Serious obstacles were encountered in addressing interconnections of iISS systems [33]. In contrast to ISS subsystems, iISS subsystems which are not ISS usher the issue of incompatibility of spaces in time domain for trajectory-based approaches, as well as insufficiency of max-type Lyapunov functions popular in ISS Lyapunov-based approaches (e.g. [38, 20]). Breakthroughs made in [32, 34, 3, 45] allowed us to use small-gain criteria as in the ISS small-gain theorem in spite of the inevitable and considerable differences between their proofs and Lyapunov constructions.

Input-to-state dynamical stability (ISDS). ISS estimate depends on the norm of the whole input and for large times it gives the following estimate for the norm of the state: $\lim_{t\to\infty} |\phi(t,x,u)| \leq \gamma(\|u\|_\infty)$. At the same time we know from Proposition 3.1.1 that if the system is ISS and if $\lim_{t\to\infty} |u(t)| = 0$, then also $\lim_{t\to\infty} |\phi(t,x,u)| = 0$ for any $x \in \mathbb{R}^n$. This means that for inputs, which are large for the small times and then decay to smaller values ISS estimate is too rough. To overcome this drawback, the notion of input-to-state dynamical stability (ISDS) has been introduced and studied in [25]. ISDS is equivalent to ISS, but it provides a less restrictive, although more sophisticated estimate with a so-called memory-fading effect for the norm of the trajectory.

6.1 ISS for other classes of systems

On the other hand, significant research efforts have been devoted to the extension of ISS theory to further classes of systems, such as discrete-time [40], impulsive [28], [17], switched [81] and hybrid [11], [16] systems.

For time-delay systems, a particular class of infinite-dimensional systems, an extensive ISS theory is available. In this context two distinct sufficient conditions of Lyapunov-type have been proposed: using ISS Lyapunov-Razumikhin functions [73] and in terms of ISS Lyapunov-Krasovskii functionals [64]. For converse Lyapunov theorems see [42]. In [43] a general small-gain theorem for abstract systems has been proved and small-gain results for finite-dimensional and time-delay systems have been provided.

6.2 Input-to-state stability and stabilizablity in infinite dimensions

Definitions

In [18, 19], ISS of nonlinear distributed parameter systems (DPS) has been studied in the framework of semigroup theory [36], [14]. In this context infinite-dimensional systems are represented in the form

$$\dot{x}(t) = Ax(t) + f(x(t), w(t)),\ x(t) \in X, w(t) \in W, \tag{6.2.1}$$

where X is a Banach space, A is the generator of a C_0-semigroup and $f : X \times W \to X$ satisfies suitable regularity conditions. We denote the set of admissible input functions by W_c endowed with a norm $\|\cdot\|_{W_c}$. Many classes of evolution PDEs, e.g. parabolic and hyperbolic PDEs, can be written in this way, [27].

System (6.2.1) is called ISS with respect to W_c, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ so that for all initial conditions $\phi_0 \in X$ and all inputs $w \in W_c$ the state of the system (6.2.1) satisfies the estimate

$$\|\phi(t, \phi_0, w)\|_X \leq \beta(\|\phi_0\|_X, t) + \gamma(\|w\|_{W_c}) \quad t \geq 0. \tag{6.2.2}$$

If there exist $\beta \in \mathcal{KL}$ so that for all initial conditions $\phi_0 \in X$ and for all $t \geq 0$ we have

$$\|\phi(t, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t)\,, \tag{6.2.3}$$

then (6.2.1) is called uniformly globally asymptotically stable at zero (0-UGAS). Other properties such as AG and GS are defined in a similar way.

Stability and ISS of linear distributed parameter systems

Problems of stability and robust stabilization of control systems are as important for distributed parameter systems as they are for ODEs. However, the theory of ISS is much less developed when it comes to PDEs. For infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t),$$

where A is the generator of a C_0-semigroup, different notions of stability such as uniform exponential stability, polynomial stability and strong stability are well understood [57, 79]. In contrast to the finite-dimensional case the notions of uniform exponential stability and global asymptotic stability are not equivalent in infinite-dimensions. Also criteria based on the location of the spectrum of the generator cannot be easily generalized to infinite dimensions.

In contrast to the maturity of classical stability theory, only a few ISS results are known for linear PDEs, but the interest in applications of ISS theory grows rapidly. To discuss the linear case, let $f(x, w) = Bw$ be a linear but not necessarily bounded operator. Then system (6.2.1) takes the form

$$\dot{x}(t) = Ax(t) + Bw(t),\ x(t) \in X, w(t) \in W. \tag{6.2.4}$$

System (6.2.4) is ISS with respect to $L_p(\mathbb{R}_+, W)$ if and only if A generates an exponentially stable C_0-semigroup and the control operator B is (infinite-time) p-admissible. Here B is called p-admissible, with $p \in [1, \infty]$, if there exist constants M_t depending on t such that

$$\left\| \int_0^t T(s)Bu(s)\, ds \right\|_X \leq M_t \|u\|_{L_p((0,t),U)}\,, \quad \forall\; u \in L_{p,loc}(\mathbb{R}_+, U),\; t \geq 0\,. \tag{6.2.5}$$

In many applications it is required that B is infinite-time p-admissible, that is, the constant M_t in (6.2.5) is independent of t [35], [77]. Both ISS and integral ISS imply 0-UGAS, i.e. exponential stability of $(T(\cdot))_{t\geq 0}$, and for exponentially stable semigroups p-admissibility and infinite-time p-admissibility of control operators are equivalent notions.

Several papers study ISS of linear systems by means of Lyapunov methods. They have been successfully applied to derive conditions for ISS of linear systems of hyperbolic equations (balance laws) [67] and of a diffusion equation with boundary inputs [7]. These methods have been also applied to the control of the poloidal magnetic flux profile in a Tokamak plasma, [8] and to the robust stabilization of a diffusion equation with time-varying distributed coefficients [6].

In [37] and [56] ISS of linear infinite-dimensional systems has been investigated via frequency-domain methods. In [37] relations between the circle-criterion and ISS have been provided and in [56] the problem of stabilization of DPS via observer-based (dynamic) feedback has been addressed.

ISS theory for nonlinear distributed parameter systems

Lyapunov methods predominate in the existing approaches to ISS theory for the nonlinear system (6.2.1). The paper [18] presents a sufficient condition for ISS of (6.2.1) in terms of ISS Lyapunov functions. Also Lyapunov small-gain theorems are presented which yield constructions of ISS Lyapunov functions for large-scale systems. On the basis of [18], ISS for impulsive infinite-dimensional systems has been developed in [19]. In [60] the ISS property of a class of semilinear parabolic equations has been studied with the help of ISS Lyapunov functions.

In [43] a general vector Lyapunov small-gain theorem for abstract control systems satisfying a weak semigroup property has been proved. This framework formed the basis for a study of stability and stabilizability of nonlinear systems in [44].

An important subclass of control systems, which forms a bridge between linear and nonlinear theory is that of bilinear systems. Such systems arise in a number of applications such as biochemical reactions or quantum-mechanical processes, [63]. Globally asymptotically stable bilinear systems possess some form of robustness, which is not as strong as ISS. This observation motivated research in the area of infinite-dimensional integral input-to-state stability (iISS) theory. We say that (6.2.1) is iISS if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ so that

$$\alpha(\|\phi(t,\phi_0,w)\|_X) \leq \beta(\|\phi_0\|_X, t) + \int_0^t \gamma(\|w(s)\|_W)ds \quad \forall \phi_0 \in X \quad \forall w \in W \quad \forall t \geq 0. \tag{6.2.6}$$

In [62] it was shown that bilinear 0-UGAS systems (6.2.1) are necessarily iISS. In addition, if A generates an analytic semigroup and X is a Hilbert space, a construction of an iISS Lyapunov function for (6.2.1) was provided, which is based on the solution of a Lyapunov equation for the system with zero input. This shows that analysis of bilinear systems requires a combination of methods from linear and nonlinear systems theory. On the other hand, these results motivated the research in iISS theory of nonlinear systems. In [?] iISS Lyapunov functions for highly nonlinear interconnected parabolic systems have been constructed by means of small-gain theorems and direct constructions of iISS Lyapunov functions for scalar parabolic equations.

Bibliography

[1] V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin. *Optimal control*. Consultants Bureau, New York, 1987.

[2] D. Angeli. Further Results on Incremental Input-to-State Stability. *IEEE Transactions on Automatic Control*, 54(6):1386–1391, 2009.

[3] D. Angeli and A. Astolfi. A tight small gain theorem for not necessarily ISS systems. *Syst. Control Lett.*, 56:87–91, 2007.

[4] D. Angeli, E. D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Transactions on Automatic Control*, 45(6):1082–1097, 2000.

[5] M. Arcak and P. Kokotović. Nonlinear observers: a circle criterion design and robustness analysis. *Automatica*, 37(12):1923–1930, 2001.

[6] F. B. Argomedo, C. Prieur, E. Witrant, and S. Bremond. A Strict Control Lyapunov Function for a Diffusion Equation With Time-Varying Distributed Coefficients. *IEEE Transactions on Automatic Control*, 58(2):290–303, 2013.

[7] F. B. Argomedo, E. Witrant, and C. Prieur. D1-Input-to-state stability of a time-varying nonhomogeneous diffusive equation subject to boundary disturbances. In *American Control Conference (ACC), 2012*, pages 2978–2983, 2012.

[8] F. B. Argomedo, E. Witrant, and C. Prieur. *Safety Factor Profile Control in a Tokamak*. Springer International Publishing, 2013.

[9] V. Arnold. *Ordinary Differential Equations*. Springer, 1992.

[10] D. S. Bernstein. *Matrix mathematics*. Princeton University Press, Princeton, NJ, second edition, 2009. Theory, facts, and formulas.

[11] C. Cai and A. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58(1):47–53, 2009.

[12] A. L. Chen, G. Q. Chen, and R. A. Freeman. Stability of Nonlinear Feedback Systems: A New Small-Gain Theorem. *SIAM Journal on Control and Optimization*, 46(6):1995–2012, 2007.

[13] E. Coddington and N. Levinson. *Teoriija obyknovennyh differentsialnyh uravneniy*. Izdatelstvo inostrannoy literatury, 1958.

[14] R. F. Curtain and H. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag, New York, 1995.

[15] S. Dashkovskiy, M. Görges, M. Kosmykov, A. Mironchenko, and L. Naujok. Modeling and stability analysis of autonomously controlled production networks. *Logistics Research*, 3(2-3):145–157, 2011.

[16] S. Dashkovskiy and M. Kosmykov. Input-to-state stability of interconnected hybrid systems. *Automatica*, 49(4):1068–1074, 2013.

[17] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok. Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods. *Nonlinear Analysis: Hybrid Systems*, 6(3):899–915, 2012.

[18] S. Dashkovskiy and A. Mironchenko. Input-to-state stability of infinite-dimensional control systems. *Mathematics of Control, Signals, and Systems*, 25(1):1–35, 2013.

[19] S. Dashkovskiy and A. Mironchenko. Input-to-State Stability of Nonlinear Impulsive Systems. *SIAM Journal on Control and Optimization*, 51(3):1962–1987, 2013.

[20] S. Dashkovskiy, B. Rüffer, and F. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010.

[21] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. On the construction of ISS Lyapunov functions for networks of ISS systems. In *Proceedings of the 17th Int. Symposium on Mathematical Theory of Networks and Systems, Kyoto, Japan*, pages 77–82, 2006.

[22] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems*, 19(2):93–122, 2007.

[23] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx, editors. *Modeling and control of complex physical systems. The port-Hamiltonian approach.* Springer-Verlag, Berlin, 2009.

[24] R. A. Freeman and P. V. Kokotovic. *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques.* Birkhäuser, Boston, MA, 2008.

[25] L. Grüne. Input-to-state dynamical stability and its Lyapunov function characterization. *IEEE Transactions on Automatic Control*, 47(9):1499–1504, 2002.

[26] W. Hahn. *Stability of motion.* Springer-Verlag, New York, 1967.

[27] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Springer-Verlag, Berlin, 1981.

[28] J. P. Hespanha, D. Liberzon, and A. R. Teel. Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica*, 44(11):2735–2744, 2008.

[29] J. P. Hespanha and A. S. Morse. Certainty equivalence implies detectability. *Systems & Control Letters*, 36(1):1–13, 1999.

[30] B. Ingalls and E. D. Sontag. A small-gain theorem with applications to input/output systems, incremental stability, detectability, and interconnections. *Journal of the Franklin Institute*, 339:211–229, 2002.

[31] A. Isidori. *Nonlinear Control Systems II.* Springer, 1999.

[32] H. Ito. State-dependent scaling problems and stability of interconnected iISS and ISS systems. *IEEE Transactions on Automatic Control*, 51:1626–1643, 2006.

[33] H. Ito. Utility of iISS in composing Lyapunov functions. In *9th IFAC Symposium on Nonlinear Control Systems*, pages 723–730, 2013.

[34] H. Ito and Z.-P. Jiang. Necessary and Sufficient Small Gain Conditions for Integral Input-to-State Stable Systems: A Lyapunov Perspective. *IEEE Transactions on Automatic Control*, 54(10):2389–2404, 2009.

[35] B. Jacob and J. R. Partington. Admissibility of Control and Observation Operators for Semigroups: A Survey. In *Current Trends in Operator Theory and its Applications*, pages 199–221. Birkhäuser Basel, 2004.

[36] B. Jacob and H. J. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces.* Springer, Basel, 2012.

[37] B. Jayawardhana, H. Logemann, and E. P. Ryan. Infinite-dimensional feedback systems: the circle criterion and input-to-state stability. *Communications in Information and Systems*, 8(4):413–444, 2008.

[38] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32(8):1211–1215, 1996.

[39] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals, and Systems*, 7(2):95–120, 1994.

[40] Z.-P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37(6):857–869, 2001.

[41] Z.-P. Jiang and Y. Wang. A generalization of the nonlinear small-gain theorem for large-scale complex systems. In *7th World Congress on Intelligent Control and Automation (WCICA)*, pages 1188–1193, 2008.

[42] I. Karafyllis. Lyapunov theorems for systems described by retarded functional differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 64(3):590–617, 2006.

[43] I. Karafyllis and Z.-P. Jiang. A vector small-gain theorem for general non-linear control systems. *IMA Journal of Mathematical Control and Information*, 28:309–344, 2011.

[44] I. Karafyllis and Z.-P. Jiang. *Stability and stabilization of nonlinear systems*. Communications and Control Engineering Series. Springer-Verlag, London, 2011.

[45] I. Karafyllis and Z.-P. Jiang. A new small-gain theorem with an application to the stabilization of the chemostat. *Int. J. Robust. Nonlinear Control*, 22(14):1602–1630, 2012.

[46] I. Karafyllis, P. Pepe, and Z.-P. Jiang. Input-to-Output Stability for Systems Described by Retarded Functional Differential Equations. *European Journal of Control*, 14(6):539–555, 2008.

[47] C. M. Kellett. A compendium of comparison function results. *Mathematics of Control, Signals, and Systems*, 26(3):339–374, 2014.

[48] H. K. Khalil. *Nonlinear systems*. Prentice Hall, New York, 2002.

[49] P. Kokotović and M. Arcak. Constructive nonlinear control: a historical perspective. *Automatica*, 37(5):637–662, 2001.

[50] A. N. Kolmogorov and S. V. Fomin. *Elementy teorii funktsiy i funktsionalnogo analiza (in Russian)*. Nauka, 1976.

[51] M. Krichman, E. D. Sontag, and Y. Wang. Input-Output-to-State Stability. *SIAM J. Control Optim*, 39(6):1874–1928, 2001.

[52] V. Lakshmikantham and S. Leela. *Differential and Integral Inequalities: Ordinary differential equations*. Academic Press, 1969.

[53] P. Leenheer, D. Angeli, and E. Sontag. Monotone Chemical Reaction Networks. *Journal of Mathematical Chemistry*, 41(3):295–314, Apr. 2007.

[54] D. Liberzon. Observer-based quantized output feedback control of nonlinear systems. In *15th Mediterranean Conference on Control and Automation*, pages 1–5, 2007.

[55] Y. Lin, E. D. Sontag, and Y. Wang. A Smooth Converse Lyapunov Theorem for Robust Stability. *SIAM Journal on Control and Optimization*, 34(1):124–160, 1996.

[56] H. Logemann. Stabilization of Well-Posed Infinite-Dimensional Systems by Dynamic Sampled-Data Feedback. *SIAM Journal on Control and Optimization*, 51(2):1203–1231, 2013.

[57] Z. H. Luo, B. Z. Guo, and Ö. Morgül. *Stability and Stabilization of Infinite Dimensional Systems With Applications*. Communications and Control Engineering Series. Springer Verlag, 1999.

[58] A. Lyapunov. *Obshchaya zadacha ob ustoichivosti dvizhenia*. Kharkov Mathematical Society, 1892.

[59] A. Lyapunov. *General Problem of the Stability Of Motion*. Control Theory and Applications Series. Taylor & Francis, 1992.

[60] F. Mazenc and C. Prieur. Strict Lyapunov functions for semilinear parabolic partial differential equations. *Mathematical Control and Related Fields*, 1(2):231–250, 2011.

[61] J. D. Meiss. *Differential Dynamical Systems*. Society for Industrial and Applied Mathematics, 2007.

[62] A. Mironchenko and H. Ito. Integral input-to-state stability of bilinear infinite-dimensional systems. In *Proc. of the 53th IEEE Conference on Decision and Control*, pages 3155–3160, 2014.

[63] P. M. Pardalos and V. Yatsenko. *Optimization and Control of Bilinear Systems*. Springer, Boston, MA, 2008.

[64] P. Pepe and Z.-P. Jiang. A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Systems & Control Letters*, 55(12):1006–1014, 2006.

[65] J. W. Polderman and J. C. Willems. *Introduction to mathematical systems theory. A behavioral approach*. Springer-Verlag, New York, 1998.

[66] L. Praly and Y. Wang. Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability. *Mathematics of Control, Signals, and Systems (MCSS)*, 9(1):1–33, 1996.

[67] C. Prieur and F. Mazenc. ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws. *Mathematics of Control, Signals, and Systems*, 24(1-2):111–134, 2012.

[68] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, 1989.

[69] E. D. Sontag. Comments on integral variants of ISS. *Systems & Control Letters*, 34(1-2):93–100, 1998.

[70] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359, 1995.

[71] E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9):1283–1294, 1996.

[72] E. D. Sontag and Y. Wang. Output-to-state stability and detectability of nonlinear systems. *Systems & Control Letters*, 29(5):279–290, 1997.

[73] A. R. Teel. Connections between Razumikhin-type theorems and the ISS nonlinear small-gain theorem. *IEEE Transactions on Automatic Control*, 43(7):960–964, 1998.

[74] A. R. Teel and L. Praly. A smooth Lyapunov function from a class-estimate involving two positive semidefinite functions. *ESAIM: Control, Optimisation and Calculus of Variations*, 5:313–367, 2000.

[75] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*. American Mathematical Society, 2012.

[76] J. Tsinias. Input to state stability properties of nonlinear systems and applications to bounded feedback stabilization using saturation. *ESAIM: Control, Optimisation and Calculus of Variations*, 2:57–87, 1997.

[77] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Basler Lehrbücher. Birkhäuser Basel, 2009.

[78] A. van der Schaft. *L_2-gain and passivity techniques in nonlinear control*. Communications and Control Engineering Series. Springer, London, second edition, 2000.

[79] J. van Neerven. *The Asymptotic Behaviour of Semigroups of Linear Operators*. Operator Theory: Advances and Applications. Birkhäuser Basel, 1996.

[80] R. Vinograd. The Inadequacy of the Method of Characteristic Exponents for the Study of Nonlinear Differential Equations (in Russian). *Matematicheskiy Sbornik*, 41(4):431–438, 1957.

[81] L. Vu, D. Chatterjee, and D. Liberzon. Input-to-state stability of switched systems and switching adaptive control. *Automatica*, 43(4):639–646, 2007.

[82] J. C. Willems. Dissipative dynamical systems part I: General theory. *Archive for Rational Mechanics and Analysis*, 45(5):321–351, 1972.

[83] J. C. Willems. Dissipative dynamical systems Part II: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45(5):352–393, 1972.

[84] J. C. Willems. Paradigms and puzzles in the theory of dynamical systems. *IEEE Transactions on Automatic Control*, 36(3):259–294, 1991.

[85] J. Zabczyk. *Mathematical control theory*. Birkhäuser, Boston, MA, 2008.